Cesàro Mean of Product Summability
Of Ordinary Differential Equations in Sequences

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Abstract— In [4], the definition of product summability method \((D,k)(C,l)\) for functions was given and some of its properties were investigated. In [2], \((D,k)(C,\alpha,\beta)\) \((k > 0, \alpha > 0, \beta > -1)\) summability for functions are defined and some of its properties were investigated. In [1], the Cesàro means and Cesàro summability were discussed for sequences. In this paper, we study some results of Cesàro mean of product summability \((D,k)(C,\alpha,\beta)\) \((k > 0, \alpha > 0, \beta > -1)\) of ordinary differential equations in sequences.

I. INTRODUCTION

Kuttner [1], introduced the summability method for functions and investigated some of its properties . Pathak [4], defined the product summability method for functions and investigated some of its properties . Mishra and Srivastava [3], introduced the summability method for functions by generalizing summability method . Mishra and Mishra [2], introduced the summability method for functions and investigated some of its properties . In this paper, we study some results of Cesàro mean of product summability \((D,k)(C,\alpha,\beta)\) \((k > 0, \alpha > 0, \beta > -1)\) of ordinary differential equations in sequences.

II. SOME RELATIONS AND DEFINITIONS

We would like to first introduce Summability method. Summability method is more general than that of ordinary convergence. If we are given a sequence \((s_n)\), we can construct a generalized sequence \(\sigma_n\), the arithmetic mean of \((s_n)\) by this sequence \((s_n)\). If \(\sigma_n\) is convergent in ordinary sense for all \(n > 0\), then we say that \((s_n)\) is summable \((C,1)\) to the sum \(s\). This \((C,1)\) is called Cesàro mean of first order.

If \(s_n \rightarrow s \Rightarrow \sigma_n = \frac{s_0 + s_1 + \ldots + s_n}{n+1} \rightarrow s\), if a sequence is convergent, it is summable by method of arithmetic mean. Also a series \(1 - 1 + 1 - 1 + \cdots\) is not convergent, but is summable to the sum \(\frac{1}{2}\). The space of summable sequences is larger than space of convergent sequences. If \(\sigma_n \rightarrow s\) as \(n \rightarrow \infty\), then we say that sequence \((s_n)\) is summable by method of arithmetic mean.

And let \(\sigma_n = \frac{a_0 + a_1 + \ldots + a_n}{n+1}\). It may happen that whereas (1) diverges, the quantities \(\frac{a_0 + a_1 + \ldots + a_n}{n+1}\) converges to a definite limit as \(n \rightarrow \infty\). For example \(\frac{1}{2} - 1 + \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \cdots\) diverges, but in this case \(s_0 = 0, s_1 = 0 = 0, s_2 = 0, s_3 = 0 \ldots \ldots \ldots \) \((s_n) = (0,0,0,0 \ldots \ldots \ldots)\). Since \(s_n = \frac{1+(-1)^n}{2}\),

\[
\sigma_n = \frac{a_0 + a_1 + \ldots + a_n}{n+1}
\]

\[
= \frac{1+(-1)^0}{2} + \frac{1+(-1)^1}{2} + \frac{1+(-1)^2}{2} + \cdots + \frac{1+(-1)^n}{2} / (n+1)
\]

\[
= \frac{(n+1)}{2} + \frac{1}{2} (1 - 1 + 1 - 1 + \cdots - (n+1) terms) / (n+1)
\]

\[
= \frac{1}{2} + \frac{1+(-1)^n}{4(n+1)}
\]

If \(n\) is even then \(\sigma_n = \frac{1}{2} - \frac{1}{2(n+1)}\) \(\rightarrow \frac{1}{2}\) as \(n \rightarrow \infty\) and if \(n\) is odd then \(\sigma_n = \frac{1}{2} \). So in either case \(\lim_{n \rightarrow \infty} \sigma_n = \frac{1}{2}\), \(\sigma_n \in C\) but \(s_n \notin \mathbb{C}\). Therefore space of summable sequences is larger than that of space of convergent sequences.

Let \(f(x)\) be any function which is Lebesgue-measurable, and that \(f : [0, + \infty) \rightarrow \mathbb{R}\), and integrable in \((0, x)\), for any finite \(x\) and which is bounded in some right hand neighbourhood of origin. Integrals of the form

\[
\int_{a}^{x} f(x) \, dx
\]

are throughout to be taken as \(\lim_{x \rightarrow \infty} \int_{0}^{x} f(x) \, dx\) being a Lebesgue integral. For any \(n > 0\), we write \(a_n(x)\) for the \(n\)th integral,

\[
a_n(x) = \frac{1}{\Gamma(n)} \int_{0}^{x} (x - y)^{n-1} a(y) dy,
\]

\(a_n(0) = a(x)\), \(a(x)\) representing dependent variable of partial differential equations of sequences. The \((C,\alpha,\beta)\) transform of \(a(t)\), which we denote by \(\tilde{\alpha,\beta}(t)\) is given by...
\[ \frac{\Gamma(\alpha + \beta + 1)}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta + 1)} \int_0^x (t-u)^{\alpha-1} u^\beta a(y) dy, \]

(\(\alpha > 0, \beta > -1\)). \hspace{1cm} (2.1)

If, for \(t > 0\), the integral defining \(\partial_{\alpha,\beta}(t)\) exists and if \(\partial_{\alpha,\beta}(t) \rightarrow s\) as \(t \rightarrow \infty\), we say that \(a(x)\) is summable \((C, \alpha, \beta)\) to \(s\), and we write \(a(x) \rightarrow s\) \((C, \alpha, \beta)\). We write

\[ g(t) = g^{(k)}(t) = k! \int_0^t \frac{x^{k-1}}{(x+t)^{k+1}} a(x) dx, \quad (k > 0) \]  

if this exists .

We also write

\[ U_{k,\alpha,\beta}(t) = k! \int_0^t \frac{x^{k-1}}{(x+t)^{k+1}} \partial_{\alpha,\beta}(x) dx, \]  

if this exists .

With the usual terminology, we say that the sequence \(a_n\) is summable ,

(I) \((D, k)\) to the sum \(s\), if \(g(t)\) tends to a limit \(s\) as \(t \rightarrow \infty\),

(II) \((D, k)\) \((C, \alpha, \beta)\) to \(s\), if \(U_{k,\alpha,\beta}(t)\) tends to \(s\) as \(t \rightarrow \infty\). When \(\beta = 0\), \((D, k)(C, \alpha, \beta)\) and \((D, k)(C, \alpha)\) denote the same method . The case \(\beta = 0\) is due to Pathak \([4]\). We know that for any fixed \(t > 0, k > 0\), it is necessary and sufficient for the convergence of \((2.3)\) that \(\int_t^\infty \frac{\partial_{a,\beta}(x)}{x^2} dx\) should converge. \((2.4)\)

If \((2.4)\) converges , write for \(x > 0\),

\[ F_{a,\beta}(x) = \int_x^\infty \frac{\partial_{a,\beta}(t)}{t^2} dt. \]

### III. MAIN RESULTS

In this section, we have following theorems for sequences analogous to \([2]\).

**Theorem 3.1:** If \(\alpha > \gamma \geq 1, k > 0\) then \(a(x) \rightarrow s(D,k)(C,\alpha-1,\beta)\), whenever \(a(x) \rightarrow s(D,k)(C,\gamma-1,\beta)\).

**Theorem 3.2:** Let \(\alpha > \gamma \geq 0, \beta > -1\), and suppose that \(a(x)\) is summable \((C, \gamma, \beta)\) to \(s\) and that \(\int_1^\infty \frac{\partial_{a,\beta}(x)}{x^2} dx\) converges . Then \(a(x)\) is summable \((D,k)(C,\alpha,\beta)\) to \(s\).

We first prove this theorem under unreasonable definition \((2.2)\). However, if the result holds with \((2.2)\), then it must also hold under the definition of \((2.3)\). This follows from the following two Lemmas.

**Lemma 3.1:** Let \(p \geq 1, \gamma > 1\). Suppose that \(f(x) \in L(0,x)\) for finite \(x > 0\). Suppose that \(f(x) \in [C, \gamma, \beta]_p\), according to the definition \((2.3)\). Define

\[ f(x) = \begin{cases} f(x), & \text{for } x \geq T \\ 0, & \text{for } x < T \end{cases} \]

Then \(\partial_{\gamma,\beta}(y)\) but with \(f(x)\) replaced by \(\tilde{f}(x)\). Then

\[ \int_0^\infty y^{p-1} \left| \frac{d}{dy} \partial_{\gamma,\beta}(y) \right|^p dy < \infty. \]

Thus \(\tilde{f}(x)\) is summable \([C, \gamma, \beta]_p\) under the definition \((2.3)\). (By a result due to Mishra and Mishra \([1]\)).

**Lemma 3.2:** Let the hypothesis be as in Lemma 3.1, and define \(\tilde{f}(x)\) as above. Let \(k > 0, \beta > -1, \alpha > 0\).

Then \([D,k][C,\alpha,\beta]\) summability of \(\{f(x)\}\) and \(\{\tilde{f}(x)\}\) are equivalent.

**Proof of Lemma 3.1:** We are given that, for some \(T > 0,\)

\[ \int_T^\infty x^{p-1} \left| \frac{d}{dx} \partial_{\alpha,\beta}(x) \right|^p dx < \infty \]  

\((3.3)\)

But since, if \((3.3)\) holds for given \(T\), it holds for any greater \(T\), it must hold for all sufficiently large \(T\). Now by standard properties of fractional integrals, and since \(\gamma > 1\), we have

\[ \int_0^T (T-u)^{\gamma-2} u^\beta |f(u)| du < \infty, \]

\((3.4)\)

for almost all \(T\) (and thus, in particular, for some arbitrary large \(T\), we may thus suppose that \(T\) should be chosen so that \((3.3)\) and \((3.4)\) hold. Since \(\partial_{\gamma,\beta}(x) = 0\) for \(x < T\), \((3.2)\) will follow if

\[ \int_T^\infty x^{p-1} \left| \frac{d}{dx} \partial_{\gamma,\beta}(x) \right|^p dx < \infty. \]
Since (3.3) holds, this will follow from Minkowski’s inequality if we prove that

$$\int_{x}^{\infty} x^{-1} \frac{d}{dx} \left\{ -\partial_{\gamma, \beta}(x) - \partial_{\gamma, \beta}(x) \right\}^p dx < \infty \quad (3.5)$$

It follows easily that

$$\frac{d}{dx} \left\{ -\partial_{\gamma, \beta}(x) - \partial_{\gamma, \beta}(x) \right\} =$$

$$\frac{\Gamma(\gamma + \beta + 1)}{\Gamma(\gamma)} \frac{1}{(x + \gamma + 1)^{\gamma + \beta + 1}} \int_{0}^{x} \left( x - y \right)^{\gamma - 2} f(y) dy$$

For relevant values of variables

$$\left| x - y \right| \leq x + y \gamma \leq x + y \gamma x$$, so that

$$\left| \frac{d}{dx} \left\{ -\partial_{\gamma, \beta}(x) - \partial_{\gamma, \beta}(x) \right\} \right| \leq$$

$$\frac{\Gamma(\gamma + \beta + 1)}{\Gamma(\gamma)} \frac{1}{(x + \gamma + 1)^{\gamma + \beta + 1}} \int_{0}^{x} \left( x - y \right)^{\gamma - 2} f(y) dy$$

If \( \gamma \leq 2 \), then for \( x > T \), we have

$$\left( x - y \right)^{\gamma - 2} \leq (T - y)^{\gamma - 2}$$, so that

$$\left| \frac{d}{dx} \left\{ -\partial_{\gamma, \beta}(x) - \partial_{\gamma, \beta}(x) \right\} \right| \leq$$

$$\frac{\Gamma(\gamma + \beta + 1)}{\Gamma(\gamma)} \frac{1}{x^{\gamma + \beta + 1}} \int_{0}^{x} \left( x - y \right)^{\gamma - 2} f(y) dy$$

$$= \text{Const.} \quad \frac{1}{x^{\beta + \gamma}}$$ \quad (3.4).

If \( \gamma \geq 2 \), then \( \left( x - y \right)^{\gamma - 2} \leq x^{\gamma - 2} \), so that

$$\left| \frac{d}{dx} \left\{ -\partial_{\gamma, \beta}(x) - \partial_{\gamma, \beta}(x) \right\} \right| \leq$$

$$\frac{\Gamma(\gamma + \beta + 1)}{\Gamma(\gamma)} \frac{1}{x^{\gamma + \beta + 1}} \int_{0}^{x} \left( x - y \right)^{\gamma - 2} f(y) dy$$

$$= \text{Const.} \quad \frac{1}{x^{\beta + \gamma}}$$.

Since \( \gamma > 1 \), (3.5) will follow in any case.

**Proof of Lemma 3.2:** We use notations as in Lemma 3.1, and write further \( \overline{U}_{k, \alpha, \beta}(y) \) for the expression corresponding to \( U_{k, \alpha, \beta}(y) \) but with \( f(x) \) replaced by \( \tilde{f}(x) \). We know that for any fixed \( y > 0, k > 0, \beta > -1, \alpha > 0 \) convergence of

$$U_{k, \alpha, \beta}(y) = ky \int_{0}^{\infty} \frac{x^{k-1}}{(x + y)^{k+1}} \partial_{\alpha, \beta}(x) dx$$

is equivalent to the convergence of

$$\int_{1}^{\infty} \frac{\partial_{\alpha, \beta}(x)}{x^2} dx$$ . Then the conclusion will follow from Minkowski’s inequality, if we show that

$$\int_{1}^{\infty} \left| U_{k, \alpha, \beta}(y) - \overline{U}_{k, \alpha, \beta}(y) \right|^p dy < \infty$$, \quad (3.6)

Where we take (3.6) as including the assertion that the integral defined by \( U_{k, \alpha, \beta}(y) - \overline{U}_{k, \alpha, \beta}(y) \) converges for all \( y > 0 \). For large \( y \), we have

$$\partial_{\alpha, \beta}(y) - \overline{\partial}_{\alpha, \beta}(y) =$$

$$\frac{\Gamma(\gamma + \beta + 1)}{\Gamma(\gamma)} \frac{1}{x^{\gamma + \beta + 1}} \int_{0}^{x} (y - x)^{\alpha - 1} f(x) dx$$ \quad (3.7)

$$= O(1) \frac{1}{y^{\alpha + \beta}} \int_{0}^{x} (y - x)^{\alpha - 1} x^\beta f(x) dx$$

$$= O(\frac{T}{y^{\alpha + \beta}}) \quad (T < y)$$. Hence the convergence of

$$ky \int_{0}^{\infty} \frac{x^{k-1}}{(x + y)^{k+1}} \partial_{\alpha, \beta}(x) \left( \partial_{\alpha, \beta}(x) - \overline{\partial}_{\alpha, \beta}(x) \right) dx$$ , follows at once by a result due to (Mishra and mishra [2]) . Now (3.6) is equivalent to

$$\int_{1}^{\infty} \left| \int_{1}^{\infty} \frac{x^{k-1}}{(x + y)^{k+2}} (x - ky) \left( \partial_{\alpha, \beta}(x) - \overline{\partial}_{\alpha, \beta}(x) \right) dx \right|^p dy < \infty$$ \quad (3.8)
Let \( T_0 \) be any sufficiently large constant. Then (3.8) will follow from Minkowski’s inequality, if we show that

\[ \int_1^\infty y^{p-1}dy \sum_{k=0}^{T_0} \frac{x^{k-1}}{(x+y)^{k+2}} (x - ky) \left[ \partial_{\alpha,\beta}(x) - \overline{\partial}_{\alpha,\beta}(x) \right] dx < \infty \]  

(3.9)

\[ \int_1^\infty y^{p-1}dy \sum_{k=0}^{T_0} \frac{x^{k-1}}{(x+y)^{k+2}} (x - ky) \left[ \partial_{\alpha,\beta}(x) - \overline{\partial}_{\alpha,\beta}(x) \right] dx < \infty \]  

(3.10)

For \( x < T_0 \), we have

\[ \int_1^\infty y^{p-1}dy \sum_{k=0}^{T_0} \frac{x^{k-1}}{(x+y)^{k+2}} (x - ky) \left[ \partial_{\alpha,\beta}(x) - \overline{\partial}_{\alpha,\beta}(x) \right] dx \]

By (3.9), we have

\[ \int_1^\infty y^{p-1}dy \sum_{k=0}^{T_0} \frac{x^{k-1}}{(x+y)^{k+2}} (x - ky) \left[ \partial_{\alpha,\beta}(x) - \overline{\partial}_{\alpha,\beta}(x) \right] dx \]

Hence (3.9) follows. By (3.7), the expression on the left of (3.10) does not exceed a constant. Thus

\[ \int_1^\infty y^{p-1}dy \sum_{k=0}^{T_0} \frac{x^{k-1}}{(x+y)^{k+2}} (x - ky) \left[ \partial_{\alpha,\beta}(x) - \overline{\partial}_{\alpha,\beta}(x) \right] dx \]

\[ = o(1) \int_1^\infty y^{p-1}dy \sum_{k=0}^{T_0} (x+y)^{-k-1} dx \]  

(3.11)

By an obvious change of variables the expression (3.11) is equal to

\[ o(1) \int_1^\infty y^{p-1}dy \int_1^\infty t^{-2} (t - y)^{-\beta-1} dt \]

The result follows.

Proof of Theorem 3.2: We divide the proof into the following cases.

Case I. \( \alpha > \gamma \)

Case II. \( \alpha = \gamma \)

Case III. \( \alpha < \gamma \)

Here we observe that Case I and II follow from case III, with the aid of Theorem 3.1.

For, if \( \alpha \geq \gamma \), Choose any \( \gamma' > \alpha \), summability \( [C,\gamma,\beta]_p \) implies summability \( [C,\gamma',\beta]_p \) by Theorem 3.1, and it follows from Case III, that this implies \( \| (D,k)(C,\alpha,\beta) \|_p \). Hence it is sufficient to consider the case III only.

Proof of Case III: Since \( f(x) \rightarrow s(C,\alpha,\beta) \) implies that \( f(x) \rightarrow s(C,\alpha',\beta) \) for \( \alpha' > \alpha > \alpha \), there is no loss of generality in considering the Case \( \gamma = \alpha + k \), \( k \) is a positive integer.

We have, by (Mishra & Mishra [2])

\[ \frac{d}{dy} U_{k,\alpha,\beta}(y) = C \int_1^\infty \frac{x^{k-1}}{(x+y)^{k+2}} (x - ky) \partial_{\alpha,\beta}(x) dx \]  

(3.12)

Now, by definition

\[ \partial_{\alpha+p,\beta}(x) = \frac{\Gamma(\alpha + \beta + 1)}{\Gamma(\alpha + p + \gamma + \beta + 1)} \frac{1}{y^{\alpha+p+\gamma}} \int_0^y \frac{(x-t)^{\alpha+p-1}t^{\gamma+\beta}}{t^{\gamma+\beta}} \partial_{\alpha,\beta}(x) dt. \]

Putting \( p=1 \) and \( \alpha = \gamma \), we see that

\[ \partial_{\alpha+1,\beta}(x) = \frac{(\alpha + \beta + 1)}{\Gamma(\alpha + p + \gamma + \beta + 1)} \frac{1}{y^{\alpha+p+\gamma}} \int_0^y \frac{t^{\alpha+p-1}t^{\gamma+\beta}}{t^{\gamma+\beta}} \partial_{\alpha,\beta}(x) dt. \]  

(3.13)

We also write \( R_{\alpha,\beta}(x) = \int_1^\infty \frac{\partial_{\alpha,\beta}(x)}{t^s} dt. \)

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It is clear that, whenever \( \int_1^{\infty} \frac{\partial_{\alpha+\beta}(x)}{x^2} \, dx \) converges, 
\( R_{\alpha,\beta}(x) \) is defined for \( x > 0 \), and that \( R_{\alpha,\beta}(x) \to 0 \) as \( x \to \infty \). It follows immediately from (3.13) that
\[
\partial_{\alpha+1,\beta}(x) = -\frac{(\alpha + \beta + 1)}{x^{\alpha+\beta+1}} \int_0^x t^{\alpha+\beta} t \, dR_{\alpha,\beta}(t) dt
\]
\[= o(x^k) \]
And hence that, for \( p \geq 1 \),
\[
\partial_{\alpha+1,\beta}(x) = o(x^k) \tag{3.14}
\]
Now by (3.12), we have

\[
\frac{d}{dy} U_{k,\alpha,\beta}(y) = c \int_0^y \frac{x^{k-\alpha-\beta-1}}{(x+y)^{k+1}} (x-ky) x^{\alpha+\beta} \partial_{\alpha,\beta}(x) dx.
\]
(3.15)

Integrating (3.15) by parts \( k \) times, we deduce with the help of (3.14) that

\[
\frac{d}{dy} U_{k,\alpha,\beta}(y) = (-1)^k c \int_0^y \frac{x^{k-\alpha-\beta-1}}{(x+y)^{k+1}} \frac{dx}{dx} \left( \frac{x^{k-\alpha-\beta-1}}{(x+y)^{k+1}} \right) dx.
\]
(3.16)

It is verified that expression in (3.16) is \( O \left( \frac{x^{k-\alpha-\beta-1}}{(x+y)^{k+1}} \right) \)
(3.17)

Let \( R(x, y) = \int_0^x t^{k-\alpha-\beta-1} \, dt \).

In fact, for fixed \( k > 0 \), we have uniformly in \( x > 0, y > 0 \),
\[
R(x, y) = o \left( \frac{x^k}{(x+y)^{k+1}} \right) \tag{3.18}
\]
This may be proved by induction on \( k \), if \( k = 0 \), we have
\[
R(x, y) = \int_0^x t^{k-\alpha-\beta-1} \frac{dt}{(t+y)^{k+1}} (t-ky) \]
hence the result is evident. Suppose that \( k \geq 1 \), and assume the result true for \( k-1 \).

Now integrating (3.16) by parts, we have
\[
\frac{d}{dy} U_{k,\alpha,\beta}(y) = \int_0^y R(x, y) \left( \frac{d}{dx} \partial_{\alpha+k,\beta}(x) \right) dx
\]
\[= \int_0^y R(x, y) \left( \frac{d}{dx} \partial_{\gamma,\beta}(x) \right) dx .
\]
Since the integrated term tends to 0 as \( \partial_{\gamma,\beta}(x) \) is bounded and \( R(x, y) \to 0 \) as \( x \to \infty \).

Now we have
\[
\left| \frac{d}{dy} U_{k,\alpha,\beta}(y) \right|^p \leq c \int_0^y \left| R(x, y) x^{p-1} \right|^p \left( \frac{d}{dx} \partial_{\gamma,\beta}(x) \right) \left( \frac{R(x, y)}{x} \right)^{\frac{1}{p}} \, dx
\]
Applying Holder’s inequality with indices \( p \) and \( \frac{p}{p-1} \), we have
\[
\left| \frac{d}{dy} U_{k,\alpha,\beta}(y) \right|^p \leq c \int_0^y \left| R(x, y) x^{p-1} \right|^p \left( \frac{d}{dx} \partial_{\gamma,\beta}(x) \right) \left( \frac{R(x, y)}{x} \right)^{\frac{1}{p}} \frac{dx}{x} \left( \frac{R(x, y)}{x} \right)^{\frac{p-1}{p}}
\]
Using (3.18) and putting \( x = t y \), we see that the expression in curly brackets-
\[ \int_0^\infty \frac{x^{k-1}}{(x+y)^{k+1}} \, dx = C \, \int_0^\infty \frac{t^{k-1}}{(1+t)^{k+1}} \, dt = \frac{C}{y}, \]

(Since the integral converges). Hence

\[
\int_0^\infty y^{p-1} \left| \frac{d}{dy} U_{\gamma,\nu,\beta}(y) \right|^p \, dy \leq \int_0^\infty x^{p-1} \left| \frac{d}{dx} \partial_{\gamma,\nu,\beta}(x) \right|^p \, dx = \int_0^\infty y^{p-1} \left| \frac{d}{dy} \partial_{\gamma,\nu,\beta}(y) \right|^p \, dy.
\]

Again using (3.18), the inner integral

\[
\leq C \int_0^\infty \frac{1}{(x+y)^{k+1}} \, dy, \quad (3.19)
\]

On putting \( y = x^1 \), the expression on the right of (3.19) is equal to

\[
C \int_0^\infty \frac{1}{(1+t)^{k+1}} \, dt = C.
\]

(Since the integral converges). Hence the result follows.

REFERENCES