LOCAL FRACTIONAL FOURIER TRANSFORM
WITH APPLICATIONS IN LOCAL FRACTIONAL DERIVATIVE: Another Approach

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Abstract—The aim of this paper is to study the Local Fractional Fourier transforms. We have proved some properties of these transforms. As an application of Local fractional Fourier transforms, we solve some fractional order differential equations with the help of Local FrFT.

Keywords: Fractional Fourier Transform, Local Fractional Fourier Transform, Local fractional derivative, fractional differential equation.

I. INTRODUCTION

The applications of fractional transforms to generalized function have been done time to time and their properties has been studied by various mathematicians. Fourier transform is a very powerful tool for problems in signal processing and other applications. The fractional Fourier transform was proposed by Namias and developed by McBride. Furthermore, it has been studied by many researchers and contributed. The fractional calculus have several applications in various fields of Mathematics as well as in real life situations, such as Abel’s integral equation, viscoelasticity, capacitor theory, conductance of biological systems [5, 7]. The idea of fractional operators, fractional derivative, fractional geometry has long back history but fractional transform has been rediscovered in quantum mechanics, optics, signal processing as well as in pattern recognition. Now a days, many linear boundary value and initial value problems in applied mathematics, mathematical physics, and engineering science are effectively solved by fractional Fourier and fractional Hartley transforms.

The classical theory of local fractional calculus introduced by Kolwankar and Gangal [3] which becomes useful tool in the areas ranging from fundamental science to engineering. The paper is organized as follows:

II. DEFINITIONS OF FRACTIONAL DERIVATIVES AND FRACTIONAL INTEGRALS

(Gr"unwald-Letnikov) The Gr"unwald-Letnikov fractional derivative of order \( \alpha \) of the function \( f(x) \) is define as [7]

\[
aD_x^\alpha f(x) = \lim_{N \to \infty} \left\{ \left( \frac{x}{N} \right)^{-\alpha} \frac{1}{\Gamma(-\alpha)} \sum_{j=0}^{N-1} \frac{\Gamma(j+1)}{\Gamma(j+1)} f(x-j\left(\frac{x-a}{N}\right)) \right\}
\]

where, \( \alpha \in \mathbb{C} \).

(Riemann-Liouville): Riemann (1953), Liouville (1832)
If \( f(x) \in C[a, b] \) and \( a < x < b \) then

\[
D_a^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^\alpha} dt, \text{ where } \alpha \in (0, 1)
\]

is called the Riemann-Liouville fractional derivative of order \( \alpha \). [7]

(M. Caputo (1967)): If \( f(x) \in C[a, b] \) and \( a < x < b \) then the Caputo fractional derivative of order \( \alpha \) is defined as follows [7]

\[
aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(\tau)}{(x-\tau)^{\alpha-n+1}} d\tau, \quad n-1 < \alpha < n
\]

III. DEFINITIONS OF LOCAL FRACTIONAL FOURIER TRANSFORM

This section is devoted for the definition of Local Fractional Fourier Transform.

The complex number \( Z = x + iy \) can also be written in the polar form as \( z = re^{i\theta} \), so that fractional order \( \alpha \) of a complex number \( Z \) can be defined as \( z^\alpha = (re^{i\theta})^\alpha \)

\[
0 < \alpha \leq 1
\]

The above definition can also have equivalent formula in the form of trigonometric function defined by the expression,

\[
Z^\alpha = (\sqrt{x^2+y^2})^\alpha (\cos(\alpha \theta) + i \sin(\alpha \theta)), \quad 0 < \alpha \leq 1
\]

In section 2, We study some definitions which are useful for further developments. The section 3, is devoted for the definition of local fractional Fourier transform. In section 4, we prove some basic properties of local fractional Fourier transforms. In the last section, we solve the Fractional Differential Equation By using fractional Fourier transforms.
Local Fractional Fourier Transform
The Local Fractional Fourier transform of function \( f(t) \) is defined as follows
\[
\hat{f}_\alpha(\xi) = \int_R f(t) e^{-2i\pi \xi t^\alpha} dt, \quad 0 < \alpha \leq 1
\]
where as \( \alpha \to 1 \), the Local fractional Fourier transform tends to be an ordinary Fourier Transform.

To find the Local fractional Fourier transform of given function \( f(t) \) at a point \( \xi \), The sufficient condition for convergence is \( \int_R |f(t)| < \infty \)

Inverse Local Fractional Fourier Transform
The Inverse Local Fractional Fourier transform of function \( \hat{f}_\alpha(\xi) \) is defined as follows
\[
f(t) = \int_R \hat{f}_\alpha(\xi) e^{2i\pi \xi t^\alpha} d\xi, \quad 0 < \alpha \leq 1
\]
where as \( \alpha \to 1 \), the Inverse local fractional Fourier transform tends to be an ordinary Inverse Fourier Transform.

Local Fractional Derivative
Let \( f : [a, b] \to \mathbb{R} \) be any function, the limit
\[
D^\alpha \pm f(x) = \lim_{x \to y \pm} \frac{d^\alpha(f(x) - f(y))}{d(\pm(x - y))^\alpha}
\]
exist and is finite, then \( f \) is said have right(left) LFD \([3]\) of order \( \alpha \) at \( x = y \).

IV. PROPERTIES OF LOCAL FRACTIONAL FOURIER TRANSFORM

In this section, we prove some basic properties of Local Fractional Fourier Transform
(i) If \( g(t) = f(at) \) where \( a > 0 \) then
\[
\hat{g}_\alpha(\xi) = \frac{1}{a} \hat{f}_\alpha \left( \frac{\xi}{a^\alpha} \right)
\]
Proof: By definition (III), we have
\[
\hat{g}_\alpha(\xi) = \int_R f(at) e^{-2i\pi \xi t^\alpha} dt, \quad 0 < \alpha \leq 1
\]
On substituting \( at = x \) in equation (1), we get
\[
\hat{g}_\alpha(\xi) = \frac{1}{a} \int_R f(x) e^{-2i\pi \xi x^\alpha} dx
= \frac{1}{a} \int_R f(t) e^{-2i\pi t(\frac{x}{a^\alpha})} dt
\]
\[
\hat{g}_\alpha(\xi) = \frac{1}{a} \hat{f}_\alpha \left( \frac{\xi}{a^\alpha} \right)
\]
(ii) If \( g(t) = f(t + \beta) \) where \( \beta \in \mathbb{R} \) then
\[
\hat{g}_\alpha(\xi) = e^{2i\pi \beta t^\alpha} \hat{f}_\alpha(\xi)
\]
Proof: By definition (III), we have
\[
\hat{g}_\alpha(\xi) = \int_R f(t + \beta) e^{-2i\pi \xi t^\alpha} dt
\]
On substituting \( t + \beta = x \) in equation (2), we get
\[
\hat{g}_\alpha(\xi) = \int_R f(x) e^{-2i\pi (x - \beta) t^\alpha} dx
= e^{2i\pi \beta t^\alpha} \hat{f}_\alpha(\xi)
\]
(iii) If \( f(t) \) is a function which vanishes at \( \pm \infty \) then
\[
\hat{f}^{(n)}_\alpha(t)(\xi) = (2i\pi \xi)^n \hat{f}_\alpha(\xi)
\]
Proof: By definition (III), we have
\[
\hat{f}^{(n)}_\alpha(t)(\xi) = \int_R f^{(n)}(t) e^{-2i\pi \xi t^\alpha} dt
\]
Applying regular integration by parts \([8]\) to the equation (3)
\[
= \int_R f^{(n)}(t)e^{-2i\pi \xi t^\alpha} dt
= e^{-2i\pi \xi t^\alpha} \int_0^\infty \frac{d}{dt} \left( e^{-2i\pi \xi t^\alpha} \int_R f^{(n)}(t) \right) dt
= (2i\pi \xi)^n \left[ \int_0^\infty \frac{d}{dt} \left( e^{-2i\pi \xi t^\alpha} \int_R f^{(n)}(t) \right) dt \right]
\]
After, continuing this process \( n \) times, we get
\[
\hat{g}_\alpha(\xi) = (2i\pi \xi)^n \hat{f}_\alpha(\xi)
\]

V. LOCAL FRACTIONAL FOURIER TRANSFORM OF SOME FUNCTIONS

Example (i): Consider the function which is define in the following way,
\[
g(t) = \begin{cases} 
D^\alpha(n + \sqrt{t}), & 0 \leq t < \infty \\
0, & \text{otherwise} 
\end{cases}
\]
with initial condition \( f(0) = \frac{1}{2} \) then from (II),(III) and (4) we get,
\[
\hat{g}_\alpha(\xi) = \int_0^\infty g(t)e^{-2i\pi \xi t^\alpha} dt
\]
\[
= \int_0^\infty \left( D^\alpha(n + \sqrt{t}) \right) e^{-2i\pi \xi t^\alpha} dt
\]
\[
= \int_0^\infty \left( \lim_{x \to 0^+} \frac{d^{0.5}}{dt^{0.5}} \sqrt{t} \right) e^{-2i\pi \xi t^\alpha} dt
\]
\[
= \int_0^\infty \left( \lim_{x \to 0^+} \frac{\sqrt{t}}{2} \right) e^{-2i\pi \xi \frac{t^{0.5}}{2}} \int_0^t (t - \tau)^{-0.5} \tau^{0.5} d\tau e^{-2i\pi \xi \frac{t^{0.5}}{2}} dt
\]
\[
= \frac{\sqrt{\pi}}{2} \int_0^\infty e^{-2i\pi \xi \frac{t^{0.5}}{2}} dt
\]
\[
= \frac{1}{4i\sqrt{\pi \xi}}
\]
Example (ii): Consider the function which is define in the following way,
\[
g(t) = \begin{cases} 
D^\alpha(\sqrt{\frac{t}{1-t}}), & 0 \leq t < \infty \\
0, & \text{otherwise} 
\end{cases}
\]
with initial condition $f(0) = 0$ and $\alpha = \frac{1}{2}$ then from (II), and (III) we get,

$$\widehat{g}_\alpha(\xi) = \int_\mathbb{R} g(t) e^{-2i\pi \xi \alpha} dt$$

$$= \int_0^\infty D^\alpha \left( \sqrt{\frac{t}{1-t}} \right) e^{-2i\pi \xi \alpha} dt$$

$$= \int_0^\infty \left( \lim_{t \to 0} \frac{d^{0.5}}{dt^{0.5}} \left( \sqrt{\frac{t}{1-t}} \right) \right) e^{-2i\pi \xi \alpha} dt$$

$$= \int_0^\infty \left( \lim_{t \to 0} \Gamma \left( \frac{1}{2} \right) \frac{d}{dt} \int_0^t (t-\tau)^{-0.5} \left( \frac{\tau}{1-\tau} \right)^{0.5} d\tau \right) e^{-2i\pi \xi \alpha} dt$$

$$= \int_0^\infty \left( \lim_{t \to 0} \frac{E(t)}{(1-t)^{\frac{1}{\sqrt{\pi}}}} \right) e^{-2i\pi \xi \alpha} dt$$

$$= \int_0^\infty \frac{1}{\sqrt{\pi}} \left( \int_0^\frac{\pi}{2} d\theta \right) e^{-2i\pi \xi \alpha} dt$$

$$= \frac{\sqrt{\pi}}{2} \int_0^\infty e^{-2i\pi \xi \alpha} dt$$

$$= \frac{1}{4i\sqrt{\pi} \xi \alpha}$$

where, $E(t) = \int_0^{\frac{\pi}{2}} \sqrt{1-t \sin^2 \theta} d\theta$

VI. CONCLUSION

(i) The local fractional Fourier transform and fractional power of complex number is investigated.

(ii) Some properties of local fractional Fourier transform are derived.

(iii) By using local fractional Fourier transform of some functions have been evaluated.

REFERENCES


