State Derivative Feedback Control Application for Pole Placement Problem

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Abstract— The pole placement is one of the most important methods for designing controller for linear systems. In this paper work a procedure for solving the pole-placement problem for a linear time-invariant single-input/single output (SISO) systems by state-derivative feedback is described. Pole placement design to place all closed-loop poles at desired locations. In this paper, we shall prove that a necessary and sufficient condition that the closed –loop poles can be placed at any arbitrary locations in the s-plane is that the system be completely state controllable. Then we shall discuss methods for determine the required state feedback gain matrix [12].

Keywords—Characteristics Polynomial, Feedback gain matrix, Pole placement, State Derivative feedback.

I. INTRODUCTION

The problem of pole placement by state feedback has a very long history with many previous studies [1-11]. However, this paper work focuses on a special feedback mechanism using only state derivatives instead of full state feedback and further reduced by reduction methods for finding reduced controller. Therefore this feedback is called reduced state derivative feedback. This pole-placement problem is always solved for controller systems if all Eigen values of the original system are non zero. Then we can place any arbitrary closed-loop poles to achieve the desired system performance. The solution procedure results a formula same as the Ackermann one. Its derivation is based on transformation of a linear SISO system into controllable canonical form by co-ordinate transformation, then solving the pole-placement problem by SDF and transforming the solution into the original co-ordinates [7-9].

II. POLE-PLACEMENT BY STATE DERIVATIVE FEEDBACK

Consider a controllable linear time-invariant SISO system with initial condition is zero

\[ \dot{x}(t) = Ax(t) + Bu(t) \]  

(1.1)

Where \( x(t) \) - State vector (n X 1)  
\( u(t) \) - Input vector

A- System matrix (n X n)  
B- Input matrix (n X 1)

The characteristics polynomial of matrix \( A \) is given by

\[ \det[sI - A] = s^n + a_{n-1}s^{n-1} + ....... + a_1s + a_0 = 0 \]  

(1.2)

Where \( a_0, a_1, \ldots, a_{n-1} \) are coefficients of characteristics polynomial.

The objective is to place the desired closed-loop poles of system using the constant state derivative feedback control.

\[ u = -k \dot{x} + r \]  

(1.3)

That enforces desired characteristics behaviour for the states and thus stabilizes the system. From equation (1.1) and (1.3) closed loop system becomes

\[ \dot{x}(t) = Ax(t) + B(-k \dot{x}(t) + r) \]  

\[ (I + BK) \dot{x}(t) = Ax(t) + Br \]  

(1.4)

In what follows it is assumed that \((I + BK)\) has a full rank matrix so that the closed-loop system is defined well otherwise the resulting system is not a state dynamic variable system [7-9]. Then it can be written as

\[ \dot{x} = (I + BK)^{-1} Ax(t) + (I + BK)^{-1} Br \]  

(1.5)

The closed- loop characteristics polynomial is given by

\[ \det[sI -(I + BK)^{-1} A] = 0 \]  

(1.6)

The design problem is to find the feedback gain matrix \( K \) such that the closed-loop poles \( \lambda_1, \lambda_2, \ldots, \lambda_n \) of the system (1.5) are assigned at the desired values.

III. TRANSFORMATION INTO PROGENIUS CANONICAL FORM FOR TIME-IN Variant

By means of the well-known technique [7-9] the original system (1.1) is transformed by the state transformation


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\[ z(t) = Q^{-1}x(t) \] (1.7)

Where \( z(t) \) - new transformed state vector (n X 1)

\( Q^{-1} \) - Transformed matrix (n X n)

Then transformed system matrix \( A_F(nXn) = Q^{-1}AQ \)

And the transformed control gain vector

\[ B_F(nX1) = Q^{-1}B \] (1.8)

Where transformation matrix is given by

\[ Q^{-1} = \begin{bmatrix} q_1 \\ q_2A \\ \vdots \\ q_nA^{n-1} \end{bmatrix} \] (1.9)

Where the row vector \( q_i \) (1 X n) is given by

\[ q_i = e^i R^{-1} \] (1.10)

Where \( e_i = [0 \ 0 \ 0 ... 1] \) - a unit vector

R- Controllability matrix and given by

\[ [B \ AB \ A^2B \ .... \ A^{n-1}B] \] (1.11)

Then transformed system given by

\[ x(t) = A_F z(t) + B_F u(t) \]

\[ z(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -a_0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ -a_n \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \] (1.12)

This is the Frobenius canonical form of the system.

This equation is easily modified into the standard state equation if \( F_n \neq -1 \)

\[ z_s = -a_0 z_1 - a_1 z_2 - ... - a_{n-1} z_n + u \] (1.16)

It is derived for the last state equation

\[ z_n = -(a_0 z_1 - (a_1 + F_1) z_2 - ... - (a_{n-1} + F_{n-1}) z_n - F_n z_n + r \] (1.17)

This means that the resulting Frobenius canonical form of the closed-loop system is

\[ z = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} r \] (1.19)

The compact form of closed loop system matrix is given by

\[ A_F = Q^{-1}(1 + BK)^{-1}AQ \] (1.20)

Now the closed loop characteristics polynomial equation is given by

\[ \det(sI - A_F) = s^n + \frac{a_1 + F_1}{1 + F_s} s^{n-1} + ... + \frac{a_n + F_n}{1 + F_s} \] (1.21)
The desired pole is given by either a desired coefficients of the characteristics polynomial or by the desired Eigen values of this polynomial. The desired characteristics equation is

$$D(s) = (s - \lambda_1)(s - \lambda_2) \ldots \ldots \ldots (s - \lambda_n) = s^n + d_{n-1}s^{n-1} + \ldots \ldots + d_1s + d_0$$

(1.22)

Where $\lambda_i$ are desired poles.

We have two cases.

**CASE I:**

If $a_0 \neq 0$, there exists the solution for any desired coefficients $d = [d_0, d_1, \ldots, d_{n-1}]$ such that $d_0 \neq 0$

$$F = \left[ d_1\left(\frac{a_0}{d_0}\right) - a_1 \right] q + \left[ d_2\left(\frac{a_0}{d_0}\right) - a_2 \right] q_1A + \ldots \ldots$$

$$\ldots \ldots + \left[ d_{n-1}\left(\frac{a_0}{d_0}\right) - a_{n-1} \right] q_{n-1}A^{n-1}$$

(1.23)

**CASE II:**

If $a_0 = 0$, there exists the solution for any desired coefficients $d = [d_0, d_1, \ldots, d_{n-1}]$ such that $d_0 = 0$

$$F = \left[ d_1 - a_1d_2 - a_2 \ldots \ldots - a_{n-1} \right]$$

(1.24)

Now from F the feedback gain matrix of original system is given by

$$K = FQ^{-1}$$

$$= \left( d_1\left(\frac{a_0}{d_0}\right) - a_1 \right) q + \left( d_2\left(\frac{a_0}{d_0}\right) - a_2 \right) q_1A + \ldots \ldots$$

$$\ldots \ldots + \left( d_{n-1}\left(\frac{a_0}{d_0}\right) - a_{n-1} \right) q_{n-1}A^{n-1}$$

On solving it, it comes

$$= \left( \frac{a_0}{d_0} \right) q_1A^{-1} [A^n + d_{n-1}A^{n-1} + \ldots + d_1A + d_0I]$$

$$= \left( \frac{a_0}{d_0} \right) q_1A^{-1} D(A)$$

This is the desired expression.

Where $D(A)$ is the desired characteristics polynomial (1.22) with system matrix.

**V. RESULT & DISCUSSION**

The consequential formula for the constant state-derivative feedback gain matrix

$$K = \left( \frac{a_0}{d_0} \right) e_n^T (AR)^{-1} D(A)$$

(1.25)

is the straight similarity of Ackermann’s formula for conventional state feedback [7-9]. The original Ackermann’s formula can be modified also for equivalent efficient numerical algorithms for computing the feedback gain matrix $K$. The same can be done for the state-derivative feedback. The resulting equivalent efficient formula based on desired coefficients of the characteristic polynomial is the following recursive procedure:

$$K = \left( \frac{a_0}{d_0} \right) \left( q_0 + \sum_{i=0}^{n-1} d_iq_i \right)$$

(1.26)

Where $q_0 = q_0A^{-1} = e_n^T (AR)^{-1}$

$q_i = q_{i-1}A$ where $i = 1, 2, \ldots, n$

If the pole-placement is based on the desired values of the poles such as $\lambda_i, i = 1, 2, \ldots, n$ instead of the coefficients $d_i, i = 1, 2, \ldots, n$ of the characteristics equation, then the constant-feedback gain matrix $K$ is given by

$$K = \frac{a_0}{\prod_{i=1}^{n} -\lambda_i} \prod_{i=1}^{n} (A - \lambda_i I)$$

$$= \frac{\det(A)}{\prod_{i=1}^{n} \lambda_i} e_n^T (AR)^{-1} \prod_{i=1}^{n} (A - \lambda_i I)$$

(1.27)
And once more utilizing the modification of (1.26) into an corresponding well-organized principles as in [7-9] the resulting recursive procedure for state- derivative constant-feedback gain matrix based on the desired values of poles is

\[ K = \frac{\det(A)}{\prod_{i=1}^{n} \lambda_i} \quad (1.28) \]

Where \( q'_i = q_i A^{-1} = e_n^T(AR)^{-1} \)

\[ q'_i = q'_{i-1}(A - \lambda_i I); i = 1, 2, \ldots, n \]

If the system matrix A of the original system (1.1) is similar then the feedback-gain matrix k is obtained from (1.25) and (1.26) as

\[ K = (d_1 - a_1)q_1 + (d_2 - a_2)q_1 A + \ldots + (d_{n-1} - a_{n-1})q_1 A^{n-2} \]

\[ = (d_1 - a_1)q_1^T(R)^{-1} + (d_2 - a_2)q_1^T(R)^{-1} A + \ldots + (d_{n-1} - a_{n-1})q_1^T(R)^{-1} A^{n-2} \]

\[ k = \sum_{i=1}^{n-1} (d_i - a_i)q'_i ; q'_1 = q_1 = e_n^T(R)^{-1} \]

\[ q'_i = q'_{i-1} A ; i = 1, 2, \ldots, n - 1 \]

The same can be derived for the case with given desired poles into equations analogous to (1.26) and (1.27). It has taken just n - 1 poles because the nth is zero due to the singularity of the system matrix A

\[ K = q'_1 \left( \prod_{i=1}^{n-1} (A - \lambda_i I) - \prod_{i=1}^{n-1} (A - \lambda_{n-1} I) \right) \quad (1.30) \]

VI. CONCLUSION

The paper concluded two type of solution for the pole placement using state derivative feedback control method.

If the system (3.1) is entirely controllable and the system matrix A is non singular, the poles of system (3.1) can be randomly located in the desired spaces with the omission of zero by the state-derivative feedback (1.3) using the constant-feedback-gain matrix K computed by one of (1.25)–(1.28).

If the system (1.1) is completely controllable and the system matrix A is singular, the poles of the system (1.1) can be randomly located in the desired places by means of the situation that at least one desired pole is zero by the state-derivative.

REFERENCES


