q-Extension of Dirichlet Averages of Generalized Fox-Wright Function

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Abstract-- The objective of the present paper is to investigate the q-Dirichlet averages of the generalized Fox-wright hypergeometric function introduced by Wright in (1935) [1,2]. Like the functions of the Mittag-Leffler type, the functions of the Wright type are known to play fundamental roles in various applications of the fractional calculus. This is mainly due to the fact that they are interrelated with the Mittag-Leffler functions through Laplace and Fourier transformations. The authors obtained a simple conceptual q-extension of these results by introducing q-integrals of multivariate type.

Mathematics Subject Classification: 33E12, 26A33, 30C45

Keywords-- q-Dirichlet averages, Fox-wright hypergeometric function.

I. INTRODUCTION

Recently, q-calculus has served as a bridge between mathematics and physics. Therefore, there is a significant increase of activity in the area of the q-calculus due to applications of the q-calculus in mathematics, statistics and physics. The majority of scientists in the world who use q-calculus today are physicists. The q-calculus is a generalization of many subjects, like hypergeometric series, generating functions, complex analysis, and particle physics. In short, q-calculus is quite a popular subject today. One of the important applications of q-calculus is in number theory where q-type of special generating functions, for instance q-Bernoulli numbers, q-Euler numbers, and q-Genocchi numbers are used. Here we define a new class of q-type of multivariate integrals (generalization of the beta integrals) called Dirichlet integrals.

The Dirichlet measures and Dirichlet integrals have been typical combination of gamma functions and the generalization of the beta integrals respectively. In the sequel we define a simple conceptual q-extension of the results, by making use of multiple q-gamma functions defined by Nishizaw [3] and Jackson’s representation of q-gamma functions.

II. DEFINITIONS AND PRELIMINARIES

q-Dirichlet Measures:

Let $b \in \mathbb{C}^k$; $k \geq 2$ and let $E = E_{k-1}$ be the standard simplex in $\mathbb{R}^{k-1}$. The q-measure denoted $d_q(u; q)$ is defined as

$$d_{\mu, q}(u; q) = \frac{1}{B_q(b)} u_1^{b_1} u_2^{b_2} \cdots u_k^{b_k} = \prod_{j=1}^{k} \Gamma_q(b_j) \Gamma_q(b_j - 1) \Gamma_q(b_j - 1) \cdots \Gamma_q(1).$$

Here $B_q(b) = B_q(b_1, b_2, \cdots, b_k) = \frac{\Gamma_q(b_1) \Gamma_q(b_2) \cdots \Gamma_q(b_k)}{\Gamma_q(b_1 + b_2 + \cdots + b_k)}$.

$q$-Dirichlet average: Let $\Omega$ be a convex set in $\mathbb{C}$ and let $z = (z_1, z_2, \cdots, z_n) \subset \mathbb{C}^n$, $n \geq 2$, and let $f$ be a measurable function on $\Omega$, then for $n = 2$, we define the q-Dirichlet measures and average as follows

$$d_{\mu, \beta, \beta'}(u; q) = \frac{f_q(\beta + \beta')}{\Gamma_q(\beta + \beta')} u^{\beta - 1} (1 - uq)^{\beta'} - 1 d_q(u).$$

$$R_k(\beta, \beta', q; x, y) = \frac{1}{B_q(\beta, \beta') \Gamma_q(\beta + \beta')} \int_0^1 [ux + (1 - uq)y]^k u^{\beta - 1} (1 - uq)^{\beta'} - 1 d_q(u).$$

Where $\beta, \beta' \in \mathbb{C}$, $\min \{R(\beta), R(\beta')\} > 0$, $x, y \in \mathbb{R}$. 
This paper is devoted to the study of the q-Dirichlet averages of the q-generalized Fox-wright function in the form

\[ \rho M_{Q} \left[ \begin{array}{c}
(a_1; A_1)(a_2; A_2) \\
(b_1; B_1)(b_2; B_2)
\end{array} \cdots \begin{array}{c}
(a_p; A_p) \\
(b_q; B_Q)
\end{array} \right] \left( \begin{array}{c}
(\beta, \beta', q; x, y)
\end{array} \right) = \int_{E_1} \rho \psi_Q \left( u \omega z \right) d_{\mu_{\beta \beta'}} \left( u ; q \right) \quad (3.1) \]

III. REPRESENTATION OF \( R_{q}(\beta, \beta', q ; x, y) \) AND \( \rho M_{Q} \) IN TERMS OF Q-REIMANN-LIOUVILLE FRACTIONAL INTEGRALS.

In this section we have deduced representations for the Dirichlet averages \( R_{q}(\beta, \beta', q ; x, y) \) and \( \rho M_{Q}(\beta, \beta', q ; x, y) \) with q-fractional integral operators.

**Theorem (3.1):** Let \( \beta, \beta' \in \text{Complex numbers} \), \( R(\beta) > 0 \), \( R(\beta') > 0 \), and \( x, y \) be real numbers such that \( x > y \) and \( 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0 \), and \( \rho M_{Q} \) and q- Reimann-Liouville fractional integrals respectively. Then the Dirichlet average of the q-Fox-wright functions is given by

\[ \frac{r_q(\beta+\beta')}{r_q(\beta)(x-y)\beta+\beta + 1} \left[ (I_{q,0+}^\rho \psi_Q \left( a_1; A_1)(a_2; A_2) \cdots (a_p; A_p) \right)(z; q) \right] \]

**Proof:** According to equation (3.1) and basic analogue of Fox-Wright function we have,

\[ \frac{1}{B_q(\beta, \beta')} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} r_q(a_j+nA_j)(1)}{\prod_{j=1}^{q} r_q(b_j+nB_j)(q;q)_n} \int_{0}^{1} [y + u(x - y)]^n u^{-\beta - 1} (1 - qu)^{-\beta - 1} d_q(u). \]

\[ = \frac{r_q(\beta+\beta')}{r_q(\beta') r(\beta)} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} r_q(a_j+nA_j)(1)}{\prod_{j=1}^{q} r_q(b_j+nB_j)(q;q)_n} \int_{0}^{1} [y + u(x - y)]^n u^{-\beta - 1} (1 - qu)^{-\beta - 1} d_q(u) \]

Putting \( u(x - y) = t q \), \(|q| < 1 \), in the above equation, we get

\[ \rho M_{Q} \left[ \begin{array}{c}
(a_1; A_1)(a_2; A_2) \\
(b_1; B_1)(b_2; B_2)
\end{array} \cdots \begin{array}{c}
(a_p; A_p) \\
(b_q; B_Q)
\end{array} \right] \left( \begin{array}{c}
(\beta, \beta', q; x, y)
\end{array} \right) \]

\[ = \frac{r_q(\beta+\beta')}{r_q(\beta')} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} r_q(a_j+nA_j)(1)}{\prod_{j=1}^{q} r_q(b_j+nB_j)(q;q)_n} \int_{0}^{1} [y + t q(x - y)]^n u^{-\beta - 1} (1 - \frac{t q}{x - y})^{-\beta - 1} d_q(t) \cdot \frac{x - y}{x - y}. \]

\[ = \frac{(x-y)^{-\beta-\beta'}}{B_q(\beta, \beta')} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} r_q(a_j+nA_j)(1)}{\prod_{j=1}^{q} r_q(b_j+nB_j)(q;q)_n} \times \int_{0}^{1} [y + t q(x - y)]^n u^{-\beta - 1} (1 - \frac{t q}{x - y})^{-\beta - 1} d_q(t) \]
\[
\int_0^{x-y} (t+yt)^n \{tq\} ^\beta \Gamma (x-y-tq)^{-1} d_q(t).
\]

or

\[
\int_0^{x-y} (t) \Gamma (x-y-tq)^{-1} d_q(t).
\]

This proves the theorem.

IV. SPECIAL CASES

In this section, we consider some particular cases of the theorem (3.1). By setting \(p = q = 1\) and \(A = 1, B = \beta\) and \(B = \alpha\), we get the well-known result reported in \([4]\) as follows

\[
\int_0^{x-y} t^{\beta-1} p\psi_q \left( \begin{array}{c} (a_j; A_j) \\ (b_j; B_j) \end{array} \right) _{1,p} (x-y-t) \beta^{-1} d_q(t)
\]

Further, by setting \(y = 0\) in equation (4.1) we get well-known result reported in \([5]\) which is as follows

\[
\int_0^{x-y} t^{\beta-1} E^\gamma_{\alpha,\beta} (y+t) (x-y-1)^{-1} dt
\]

In particular, when \(\beta + \beta' = \gamma\), we get

\[
\int_0^{x-y} t^{\beta-1} E^\gamma_{\alpha,\beta} (y+t) (x-y)^{-1} dt
\]

V. CONCLUSION

Here we conclude with the remark that the results and the operators proved in this paper appear to be new and likely to have useful applications to a wide range of problems of mathematics, statistics and physical sciences.

REFERENCES


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