Analytical Study of Projective Tensor Product

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Abstract---This Paper Presents the Study of the Projective Tensor Product. By defining the Projective topology $\pi$ on Locally convex spaces $E$ and $F$; $U$ & $V$ be the closed absolutely convex neighbourhoods of $O$ in $E$ and $F$ respectively, forming the set $\Gamma(U \otimes V)$ = absolutely convex hull of $U \otimes V$ in $E \otimes F$, it is proved in this paper that the projective topology $\pi$ is the finest locally convex topology on $E \otimes F$ for which the Canonical mapping $\psi : E \otimes F \to E \otimes F$ is continuous.

Keywords--- Projective topology, Locally convex spaces, Absolutely convex hull, Canonical mapping.

I. INTRODUCTION

Halub(1) and Kothe(2,3) are the pioneer worker of the present area. In fact, the present work is the extension of work done by Tomiyama(6), Srivastava et al. (4), Srivastava et al. (5) and Srivastava et al. (7). In this paper we have studied analytically about Projective Tensor Product.

Here, we use the following definitions and fundamental ideas:

Definition - I : Let $E$ and $F$ be locally convex spaces, and let $U$ & $V$ be the closed absolutely convex neighbourhoods of $O$ in $E$ and $F$ respectively, forming the set $\Gamma(U \otimes V)$ = absolutely convex hull of $U \otimes V$ in $E \otimes F$, ($E \otimes F$ is denoted as tensorial product of $E$ & $F$).

Definition - II : If $\{U\}$ and $\{V\}$ are neighbourhood bases in $E$ and $F$ respectively with $U$, $V$ closed absolutely convex, then the family $\{\Gamma(U \otimes V)\}$ is a neighbourhood basis of a locally convex topology on $E \otimes F$.

This topology is called the projective topology on $E \otimes F$ and is denoted as $E \otimes F$.

Proposition 1: Let $p(x)$ and $q(y)$ be the semi-norms defined by $U$ and $V$ respectively. The set $\Gamma(U \otimes V)$ is absorbing and thus defines a semi-norm. The semi-norm of $\Gamma(U \otimes V)$ is given by

$$p \otimes q(Z) = \inf_{i=1}^{n} p(x_i) q(y_i)$$

Where the infimum is taken over all representations $Z = \sum_{i=1}^{n} x_i \otimes y_i$ in $E \otimes F$.

Proof : First we show $\Gamma(U \otimes V)$ is absorbing. Let $x_i \not\in U$ if $p(x_i) \neq 0$

and $y_i \not\in V$ if $q(y_i) \neq 0$

also $p(x_i) = 0$ iff $\rho.x_k \not\in U$ all $\rho > 0$ and $q(y_i) = 0$ iff $\rho.y_j \not\in V$ all $\rho > 0$. So we may write

$$Z = \sum_{i=1}^{n} x_i \otimes y_i$$

$$= \sum_{i=1}^{n} p(x_i) q(y_i) \left[ \frac{x_i}{p(x_i)} \otimes \frac{y_i}{q(y_i)} \right]$$

$$+ \delta \sum_{k} p(x_k) q(y_k) \left[ \frac{x_k}{\delta} \otimes \frac{y_k}{q(y_k)} \right]$$

$$+ \delta \sum_{j} p(x_j) q(y_j) \left[ \frac{x_j}{\delta} \otimes \frac{y_j}{\delta} \right]$$

$$+ \delta^2 \sum_{m} \left[ \frac{x_m}{\delta} \otimes \frac{y_m}{\delta} \right].$$
In each of the four terms in the sum representing \( Z \), the quantity in the brackets \([ \ ]\) is in \( \Gamma (U \otimes V) \). Given \( \varepsilon > 0 \), we may choose \( \delta \) sufficiently small so that

\[
(*) : \quad Z \in \left( \sum_{i=1}^{n} p(x_i) q(y_i) + \varepsilon \right) \Gamma (U \otimes V).
\]

So \( \Gamma (U \otimes V) \) is absorbing.

Now \( \Gamma (U \otimes V) \) is absorbing convex also, so it defines a semi-norm \( r(Z) \) on \( E \otimes F \). We now show \( r(Z) = p \otimes q(Z) \).

(i) \( r(Z) \subseteq p \otimes q(Z) \). \( r(Z) \) is defined by

\[
r(Z) = \inf \lambda \varepsilon, \quad Z \in \lambda \Gamma (U \otimes V).
\]

By \( (*) \) above

\[
r(Z) \leq \inf \sum p(x_i) q(y_i) + \varepsilon = p \otimes q(Z) + \varepsilon,
\]

\( \varepsilon \) arbitrary yields \( r(Z) \leq p \otimes q(Z) \).

(ii) \( p \otimes q(Z) \leq r(Z) \), suppose \( Z \in \lambda \Gamma (U \otimes V) \).

Then \( Z = \sum \alpha_k (x'_k \otimes y'_k) \) with \( p(x'_k) \leq 1, q(y'_k) \leq 1, \sum |\alpha_k| \leq \lambda \) and \( \alpha_k \geq 0 \). For this particular representation of \( Z \), we see

\[
\sum p (\alpha_k x'_k) q(y'_k) = \sum |\alpha_k| \leq \lambda.
\]

So,

\[
p \otimes q(Z) = \inf \sum p(x_i) q(y_i) \leq \lambda.
\]

This is true for every \( \lambda \) with \( Z \in \lambda \Gamma (U \otimes V) \).

Thus \( p \otimes q(Z) \leq \inf \lambda \varepsilon, \quad Z \in \lambda \Gamma (U \otimes V) = r(Z) \).

This completes the proof.

**Proposition 2**: The projective tensor product \( E \otimes F \) of two \( \pi \) normed space \( E \), \( p \) and \( F \), \( q \) is a normed space with norm \( p \otimes q \).

If \( E \) and \( F \) are metrizable locally convex spaces with semi-norms \( p_i \leq p_2 \leq \ldots \quad \) and \( q_i \leq q_2 \leq \ldots \), respectively, then \( E \otimes F \) is metrizable with defining semi-norms \( p_1 \otimes q_1 \leq p_2 \otimes q_2 \leq \ldots \).

**Proof**: Follows immediately from Proposition 1 and definition 2.

II. MAIN RESULT

**Theorem**: The projective topology \( \pi \) is the finest locally convex topology on \( E \otimes F \) for which the canonical map \( \psi : E \times F \rightarrow E \otimes F \) is continuous.

**Proof**: \( \psi \) is continuous with respect to \( \pi \) since \( \psi (U \times F) = U \otimes F \subseteq \Gamma (U \otimes V) \). Now let \( \tau \) be any topology on \( E \otimes F \) for which \( \psi \) is continuous and let \( W \) be an absolutely convex closed \( \tau \) neighbourhood of \( 0 \). Then there exist \( U, V \) with \( \psi (U \times V) = U \otimes V \subseteq W \). Since \( W \) is absolutely convex, \( \Gamma (U \otimes V) \subseteq W \). So \( \pi \) is finer than \( \tau \).

This completes the proof of the theorem.

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