Some Results on Fuzzy Mappings for Rational Expressions

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Abstract-- The present paper deals with establishment of some fixed point and common fixed point results for fuzzy mappings over a complete metric space. We are including the rational expressions in these results.

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I. INTRODUCTION

The concept of fuzzy sets was introduced by Zadeh [21] in 1965. After that a lot of work has been done regarding fuzzy sets and fuzzy mappings. The concept of fuzzy mappings was first introduced by Heilpern [10], he proved fixed point theorem for fuzzy contraction mappings which is a fuzzy analogue of the fixed point theorem for multi valued mappings of Nadler [15]. Vijayaraju and Marudai [19] generalized the Bose and Mukherjee's [2] fixed point theorems for contractive types fuzzy mappings. Marudai and Srinivavan [14] derived the simple proof of Heilpern's [10] theorem and generalization of Nadler's [15] theorem for fuzzy mappings.

Bose and Sahini [3], Butnariu [4, 5, 6], Chang and Huang [7], Chang [8], Chitra [9], Som and Mukharjee [18] studied fixed point theorems for fuzzy mappings.


Recently, Rajendran and Balasubramanian [17] worked on fuzzy contraction mappings.

More recently Vijayaraju and Mohanraj [20] obtained some fixed point theorems for contractive type fuzzy mappings which are generalization of Beg and Azam [1], fuzzy extension of Kirk and Downing [11] and which obtained by the simple proof of Park and Jeong [16].

In the present paper we are proving some fixed point and common fixed point theorems in fuzzy mappings containing the rational expressions.

II. PRELIMINARIES

To prove the results we need following definitions and assumptions:

Fuzzy Mappings: Let X be any metric linear spaces and d be any metric in x. A fuzzy set in X is a function with domain X and values in [0,1]. If A is a fuzzy set and x \( \in \) X, the function value A(x) is called the grade of membership of x in A. The collection of all fuzzy sets in X is denoted by \( \mathcal{F}(x) \).

Now we distinguish from the collection \( \mathcal{F}(x) \) a subcollection of approximate quantities, denoted W(x).

Definition 2.1

A fuzzy subset A of X is an approximate quantity iff its \( \alpha \)-level set is a compact subset (non fuzzy) of X for each \( \alpha \in [0,1] \), and \( \sup_{x \in X} A(x) = 1 \).

\( A = \{ x : A(x) \geq \alpha \} \) if \( \alpha \in [0,1] \),

\( A = \{ x : A(x) > \alpha \} \), whenever \( B \) is clover of \( B \)

Now we distinguish from the collection F(x) a sub collection of approximate quantities, denoted W(x).

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When \( A \in W(x) \) and \( A(x_0) = 1 \) for some \( x_0 \in W(x) \), we will identify A with an approximation of \( x_0 \). Then we shall define a distance between two approximate quantities.

Definition 2.2 Let A, B \( \in W(x) \), \( \alpha \in [0,1] \), define

\[ p_\alpha(A, B) = \inf_{x \in A, y \in B} d(x, y), D_\alpha(A, B) = \text{dist}(A, B), d(A, B) = \sup_\alpha D_\alpha(A, B) \]

where dist is Hausdorff distance.
The function $p_\alpha$ is called $\alpha$-spaces, and a distance between $A$ and $B$. It is easy to see that $p_\alpha$ is non decreasing function of $\alpha$. We shall also define an order of the family $W(x)$, which characterizes accuracy of a given quantity.

**Definition 2.3** Let $A, B \in W(x)$. An approximate quantity $A$ is more accurate then $B$, denoted by $A \sqsubseteq B$, iff $A(x) \leq B(x)$, for each $x \in X$.

Now we introduce a notion of fuzzy mapping, i.e. a mapping with value in the family of approximate quantities.

**Definition 2.4** Let $X$ be an arbitrary set and $Y$ be any metric linear space. $F$ is called a fuzzy mapping iff $F$ is mapping from the set $X$ into $W(Y)$, i.e $F(x) \in W(Y)$ for each $x \in X$.

A fuzzy mapping $F$ is a fuzzy subset on $X \times Y$ with membership function $F(x,y)$ ,where function value $F(x,y)$ is grade of membership of $y$ in $F(x)$.

Let $A \in F(X), B \in F(Y)$ the fuzzy set $F^{-1}(B)$ in $F(X)$, is defined as

$$F^{-1}(B)(x) = \sup_{y \in Y} (F(x,y) \cap B(y)) \text{ where } x \in X$$

First of all we shall give here the basic properties of $\alpha$-space and $\alpha$-distance between some approximate quantities.

**Lemma 3.1:** Let $\alpha \in X$, $A \in W(X)$, and $(x)$ be a fuzzy set with membership function equal a characteristic function of set $(x)$. If $(x)$ is subset of a then $p_\alpha(x,A) = 0$ for each $\alpha \in [0,1]$.

$[10]$ called a fuzzy mapping from the set of X into a family $W(X) \subset I^1$ defined as $A \in W(X) \text{ if and only if } A(x) \text{ is compact and convex in } X \text{ for each } x \in X \text{ and } \sup \{ A(x;x) \} = 1$. In this context we give the following definitions.

$$A_\alpha = \{x: A(x) \geq \alpha \} \text{ if } \alpha \in [0,1] \text{ and } A_0 = \{x: A(x) > \alpha \} \text{, Where } B \text{ denotes the clouser of (non fuzzy set) } B$$

**Remark 2:** Notice $[x_0] = \{x\}, 1 - [x]\} \text{ will be denoted by } [x_0]$. In particular $[x] = \{[x] \}$. 

**Definition 2.5** [15]. An intuitionist fuzzy set (i-fuzzy set) $A$ of $X$ is an object having the form $A=\{A^1, A^2\}$, where $A^1, A^2 \in I^1$ and $A^1(x)+A^2(x) \leq 1$ for each $x \in X$.

We denote by IFS(X) the family of all i-fuzzy sets of $X$.

**Definition 2.6** [14] Let $x_a$ be a fuzzy point of $X$. We will say that $\{x_0, 1-x_a\}$ is an i-fuzzy point of $x$ and it will be denoted by $[x_a]$. In particular $[x] = \{[x], 1-[x]\}$.

**Definition 2.7** [15] Let $A, B \in IFS(X)$. Then $A \sqsubseteq B$ if and only if $A^1 \subset B^1$ and $B^2 \subset A^2$.

**Definition 2.8** [6] Let $X$ be a metric space and $\alpha \in [0,1]$. Consider the following family

$$\Phi_\alpha(X) = \{A \in I^1 : A \sqsubseteq W(X) \}\text{, it is clear that } \alpha \in I^1, W(X) \supset \Phi_\alpha(X)$$

**Remark 2:** Notice $\Phi_{0.5} = \{A \in I^1 : A \sqsubseteq W(X) \}$.

Now we define the family if i-fuzzy sets of $X$ as follows:

$$W_\alpha(X) = \{A \in I^1 : A \sqsubseteq W(X) \}$$

**III. MAIN RESULTS**

**Theorem 1:** Let $\alpha \in [0,1]$ and $(X, d)$ be a complete metric space. Let $F$ be continuous fuzzy mapping from $X$ into $W_\alpha(X)$ satisfying the following condition:
There exists $K, \varepsilon \in (0, 1)$ such that
\[
D_{\alpha} \left( F(x), F(y) \right) \leq K(M(x, y)), \quad \text{For all } x, y \in X \text{ with } x \neq y, \text{ and}
\]
\[
M(x, y) = \max \left\{ d(x, y), p_{\alpha}(x, Fx), p_{\alpha}(y, Fy), p_{\alpha}(x, Fx), \frac{d(x, y) + p_{\alpha}(x, Fx)}{1 + d(x, y) p_{\alpha}(x, Fx)} \right\}
\]

Then there exists $x \in X$ such that $x_n$ is a fixed fuzzy point of $F$ \iff $x_0, x_1 \in X$ such that
\[
x_1 \in F(x_0), \quad \text{with } \sum_{n=1}^{\infty} k^n d(x_0, x_1) < \infty.
\]
In particular if $\alpha = 1$ then $x$ is a fixed point of $F$.

**Proof:** If there exists $x \in X$ such that $x_n$ is fixed fuzzy point of $F$, i.e. $x_n \in F(x)$ then $\sum_{n=1}^{\infty} k^n d(x_n, x_0) = 0$. Let $x_0 \in \mathcal{K}$ and suppose that there exists $x_1 \in F(x_0) \psi$ such that $\sum_{n=1}^{\infty} k^n d(x_0, x_1) < \infty$.

\[
= K \max \left\{ d(x_n, x_{n-1}), p_{\alpha}(x_n, Fx_n), p_{\alpha}(x_{n-1}, Fx_{n-1}), p_{\alpha}(x_n, Fx_{n-1}), \frac{d(x_n, x_{n-1}) + p_{\alpha}(x_n, Fx_{n-1})}{1 + d(x_n, x_{n-1}) p_{\alpha}(x_n, Fx_{n-1})}, \frac{p_{\alpha}(x_n, Fx_n) + p_{\alpha}(x_{n-1}, Fx_{n-1})}{1 + p_{\alpha}(x_n, Fx_n)p_{\alpha}(x_{n-1}, Fx_{n-1})} \right\}
\]

Since $(F(x_n))$ is a nonempty compact subset of $X$, then there exists $x_2 \in (F(x_n))_{\psi}$ such that
\[
d(x_1, x_2) = p_{\alpha}(x_1, Fx_1) \leq D_{\alpha}(F(x_0), F(x_1))
\]
By induction we construct a sequence $\{x_n\}$ in $X$ such that
\[
x_n \subseteq (F(x_n))_{\psi},
\]
and $d(x_n, x_{n+1}) \leq D_{\alpha}(F(x_n), F(x_{n-1})$. Since $K$ is given to be the non-decreasing, so
\[
d(x_n, x_{n+1}) \leq K(M(x, y))
\]

If $\max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$
Then \( d(x_n, x_{n+1}) \leq K \{d(x_n, x_{n+1}) \} \leq d(x_n, x_{n+1}), \) which is contradiction

Therefore \( d(x_n, x_{n+1}) \leq Kd(x_{n-1}, x_n) \)

\[
= K(p_\alpha d(x_{n-1}, F(x_{n-1})) \leq K[D_\alpha (F(x_{n-1}), F(x_{n-2}))] \leq K(Kd(x_{n-1}, x_{n-2})) \ldots \ldots \ldots K^n d(x_0, x_1)
\]

\[
\Rightarrow d(x_n, x_{n+m}) \leq d(x_n, x_{n+1}) + \ldots + d(x_{n+m-1}, x_{n+m})
\]

\[
\leq K^n d(x_0, x_1) + \ldots + K^{n+m-1} d(x_0, x_1)
\]

\[
= \sum_{k=n}^{k=n+m-1} K^k d(x_0, x_1)
\]

Since \( \sum_{n=1}^{\infty} K^n d(x_0, x_1) < \infty \) it follows that there exists \( u \) such that \( d(x_n, x_{m+n}) < u \in X. \)

Therefore the sequence \( \{x_n\} \) is a Cauchy sequence in \( X. \) So by completeness of \( X, \{x_n\} \) converges to \( x \in X. \) By the help of lemma 1 and 2 we have

Consequently, \( p_\alpha (x, F(x)) = 0, \) and by lemma 1 \( x_\alpha \subseteq F(x) \)

Clearly \( x_\alpha \) is a fixed fuzzy point of the fuzzy mapping \( F \) over \( X. \) In particular if \( \alpha = 1 \) then \( x \) is a fixed point of \( F. \)

Now we will generalize this theorem for common fixed point.

**I.** \( F(X) \subseteq S(X) \cap T(X) \)

**II.** \( \{S, F\} \) and \( \{T, F\} \) are \( R- \) weakly commuting mappings.

**III.** \( D_\alpha (Fx, Fy) \leq K [M(x, y)] \forall x, y \in X \) with \( x \neq y, \) where \( M(x, y) \) is defined as

\[
M(x, y) = K \max \left\{ \begin{array}{c}
D_\alpha (Sx, Ty), D_\alpha (Sx, Fx), D_\alpha (Ty, Fy), D_\alpha (Fx, Ty), \\
\frac{D_\alpha (Sx, Ty) + D_\alpha (Fx, Ty)}{1 + D_\alpha (Sx, Ty) D_\alpha (Fx, Ty)}, \\
\frac{D_\alpha (Sx, Ty) + D_\alpha (Fx, Ty)}{1 + D_\alpha (Sx, Fx) D_\alpha (Fx, Ty)}
\end{array} \right\}
\]

Where \( K \) is non decreasing function such that \( K : [0, \infty) \rightarrow [0, \infty). \)

\( K(0) = 0 \) and \( K(t) < t \forall t \in (0, \infty), \) then \( \exists x \in X \) such that \( x_\alpha \) is common fixed fuzzy point

of \( S, T \) and \( F \) if and only if \( x_0, x_1 \in X \) such that \( \sum_{n=1}^{\infty} K^n d(x_0, x_1) < \infty. \) In particular if \( \alpha = 1, \)

then \( x \) is common fixed point of \( S, T \) and \( F. \)
Proof: Let for \( x_0 \in X \) there exists \( x_1 \) and \( x_2 \) such that \( x_1 \in (S(x_0))_\alpha \subset (F(x_0))_\alpha \), and \( x_2 \in (T(x_1))_\alpha \subset (F(x_1))_\alpha \). By induction one can construct a sequence \( \{ x_n \} \) in \( X \) such that

\[
x_{2n+1} \in \left( S x_{2n+1} \right)_\alpha \subset \left( F x_{2n} \right)_\alpha .
\]

And \( x_{2n+2} \in \left( T x_{2n+2} \right)_\alpha \subset \left( F x_{2n+1} \right)_\alpha .
\)

Since \( K \) is given to be non-decreasing. So

\[
d(x_n, x_{n+1}) \leq D_\alpha \left( F(x_{n-1}), F(x_n) \right) \leq K \cdot M(x_{n-1}, x_n)
\]

\[
= K \cdot \text{Max} \left\{ D_\alpha (S x_{n-1}, T x_n), D_\alpha (S x_{n-1}, F x_{n-1}), D_\alpha (T x_n, F x_n), D_\alpha (F x_{n-1}, T x_n), D_\alpha (S x_{n-1}, T x_n) + D_\alpha (F x_{n-1}, T x_n), D_\alpha (S x_{n-1}, F x_{n-1}) + D_\alpha (F x_{n-1}, T x_n) \right\}
\]

\[
= K \cdot \text{Max} \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_n, x_{n+1}) \right\}
\]

\[
= K \cdot \text{Max} \left\{ d(x_n, x_{n+1}), d(x_n, x_{n+1}) \right\} = d(x_n, x_{n+1})
\]

Then \( d(x_n, x_{n+1}) \leq K \cdot \text{Max} \{ d(x_n, x_{n+1}) \} \leq d(x_n, x_{n+1}), \text{which is a contradiction} .
\]

Therefore \( d(x_n, x_{n+1}) \leq Kd(x_{n-1}, x_n) \)

\[
= KD_\alpha \left( F(x_{n-2}), F(x_{n-1}) \right)
\]

\[
\leq K^2 d(x_{n-1}, x_{n-2})
\]

\[
\begin{align*}
\cdots \\
\leq K^n d(x_0, x_1)
\end{align*}
\]

\[
\Rightarrow d(x_n, x_{n+m}) \leq d(x_n, x_{n+1}) + \cdots + d(x_{n+m-1}, x_{n+m}) \leq K^n d(x_0, x_1) + \cdots + K^{n+m-1} d(x_0, x_1)
\]

\[
= \sum_{j=0}^{n+m-1} K^j d(x_0, x_1)
\]

Since \( \sum_{n=1}^{\infty} K^n d(x_0, x_1) < \infty \) it follows that there exists \( u \) such that \( d(x_n, x_{n+m}) < u \in X \).

Therefore the sequence \( \{ x_n \} \) is a Cauchy sequence in \( X \).

So by completeness of \( X \), \( \{ x_n \} \) converges to \( x \in X \) and \( (S x_{2n+1})_\alpha \subset (T x_{2n+2})_\alpha \) also converges on \( X \).

Since \( \{ S, F \} \) and \( \{ T, F \} \) are \( R \)-weakly commuting mappings. So

\[
p_\alpha (x, F(x)) \leq d(x, x_n) + p_\alpha (x_n, F(x)) \leq d(x, x_n) + D_\alpha (F(x_{n-1}), F(x)) \leq d(x, x_n) + K d(x_{n-1}, x)
\]

Consequently, \( p_\alpha (x, F(x)) = 0, \text{and by lemma 1} \)

\[
x_\alpha \subset F(x)
\]
Clearly $x_\alpha$ is a common fixed fuzzy point of the fuzzy mapping $F$, $S$ and $T$ over $X$. In particular if $\alpha = 1$ then $x$ is a common fixed point of $F$, $S$ and $T$.

IV. CONCLUSION

Fuzzy set theory and Fuzzy fixed point theory has numerous applications in applied sciences and engineering such as neural network theory, stability theory, mathematical programming, modeling theory, medical sciences (medical genetics, nervous system), image processing, control theory, communications etc. As a result, fuzzy fixed point theory has become an area of interest for specialists in fixed point theory. In this paper we are proving some fixed point and common fixed point theorems in fuzzy mappings containing the rational expressions. The result can be extended for more number of fuzzy continuous mappings.

REFERENCES