An Iterative method for Solving the Container Crane Constrained Optimal Control Problem Using Chebyshev Polynomials

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Abstract—In this paper, a computational method for solving constrained nonlinear optimal control problems is presented with an application to the container crane. The method is based on Banks' et al. iterative approach, in which the nonlinear system state equations are replaced by a sequence of time-varying linear systems. Therefore, the constrained nonlinear optimal control problem can be converted into sequence of constrained time varying linear quadratic optimal control problems. Combining this iterative approach with parameterization of the state variables using Chebyshev polynomials will result in converting the hard constrained nonlinear optimal control problem into sequence of quadratic programming problems. To show the performance and the behavior of this method compared with other known approaches, we apply it on a practical problem namely the container crane problem and the simulation results are presented and compared with other methods.

Keywords—Banks' Iterative Technique, Chebyshev polynomials, Constrained nonlinear quadratic optimal control problem, Container Crane, State parameterization.

I. INTRODUCTION

One of the widely used methods to solve the constrained nonlinear optimal control problems is the direct method, which is based on replacing the original problem by a finite dimension mathematical programming problem [1-7]. The direct methods are implemented using the discretization or the parameterization methods or both. In its turn, the nonlinear mathematical programming problem can then be solved using different methods. One of the popular methods that are used to handle the nonlinear mathematical programming problem is the sequential quadratic programming method [8] which replaces the nonlinear mathematical programming problem by a sequence of quadratic programming problems.

Jaddu [9-11] proposed a method that is based on state parameterization via Chebyshev polynomials combined with quasilinearization to handle the constrained nonlinear quadratic optimal control problems and to convert the constrained nonlinear optimal control problem into sequence of quadratic programming problems.

Also recently Jaddu and Majdalawi [12, 13] proposed new method to solve the nonlinear optimal control problem that is based on Banks' et al [14-18] iterative technique along with the state parameterization using Chebyshev and Legendre polynomials. This iterative method is based on replacing the nonlinear dynamic state equation into sequence of linear time- varying state equations. Therefore, the nonlinear optimal control problem is replaced by sequence of time- varying linear quadratic optimal control problems.

In this paper we extend our work in [12, 13] to handle nonlinear optimal control problems subject to terminal state constraints, state variables and control variables saturation constraints. After applying the iterative technique the resulted time-varying linear optimal control problems are converted into quadratic programming problem by parameterizing the state variables via Chebyshev polynomials.

In addition, the proposed method is tested by applying it to difficult constrained nonlinear optimal control problem, namely the container crane problem to minimize the swing angle during the transfer of a container from a ship to a truck. The results of the method are compared with other results obtained using other methods.

II. PROBLEM STATEMENT

The optimal control problem treated in this paper can be stated as follows: Find an optimal controller \( u^*(t) \) that minimizes the following performance index

\[
J = \int_0^{t_f} (x^T Q x + u^T R u) dt
\]  

(1)

subject to the following constraints: Nonlinear dynamic system state equations and initial conditions

\[
\dot{x} = f(x(t)) + g(x(t)) \ u(t) \quad x(0) = x_i
\]  

(2)

and terminal state constraints

\[
G x(t_f) = x_f
\]  

(3)

and saturation constraints on the state and control variables
Where $Q$ is an $n \times n$ positive semidefinite matrix, $R$ is an $m \times m$ positive definite matrix, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector, $x_0 \in \mathbb{R}^n$ is the initial condition vector, $G$ is $q \times n$ matrix, $X_{\text{max}}, X_{\text{min}}, U_{\text{min}}$ and $U_{\text{max}}$ are constant vectors of appropriate dimension, $t_f$ is the fixed final time and $x_f$ is a known final state vector. We will assume that: $m \leq n$.

The constrained nonlinear optimal control problem (1)-(4) is solved by applying iterative method to converting it into a sequence of constrained time-varying linear quadratic optimal control problems. Then, each of these problems is solved by transforming it into quadratic programming problem by using state parameterization via Chebyshev polynomials.

III. ITERATIVE TECHNIQUE

This technique was developed by Banks et al. [14-18]. In this technique, the constrained nonlinear optimal control problem (1)-(4) can be transformed into an equivalent sequence of constrained time-varying linear quadratic problems.

Applying the iterative technique to the optimal control problem described in (1)-(4), the following sequence of constrained time-varying linear quadratic optimal control problems can replace the original problem in (1)-(4):

\[
\begin{align*}
\text{Minimizes} & \quad J^{[i]} = \int_0^{t_f} (x(t)^T Q x(t) + u(t)^T R u(t)) dt \\
\text{subject to} & \quad \text{linearized state equations and initial conditions} \\
& \quad \dot{x}^{[i]} = A(x^{[i]}(t)) x^{[i]} + B(x^{[i]}(t)) u^{[i]}, \quad x^{[i]}(0) = x_0 \\
& \quad \text{and to the following terminal constraints} \\
& \quad G(x(t_f)^{[i]}) = x_f \\
& \quad X_{\text{min}} \leq x(t)^{[i]} \leq X_{\text{max}}, \quad U_{\text{min}} \leq u(t)^{[i]} \leq U_{\text{max}}
\end{align*}
\]

where $x^{[0]}(t) = x_0$.

IV. PROBLEM REFORMULATION

To convert each of the constrained time-varying linear optimal control problems (5)-(8) into a quadratic programming problem, some state variables are approximated by a finite length Chebyshev series with unknown parameters [9]. Then, the remaining state and control variables are determined as a function of the unknown parameters of the approximated state variables from the state equations (6). These approximations are used to approximate the remaining constraints, namely, initial conditions, terminal state constraints and state and control saturation constraints. For details of the state parameterization the reader is advised to consult [9].

Chebyshev polynomials are defined on the interval $\tau \in [-1,1]$. Therefore, it is necessary to transform the time interval of the original problem $t \in [0,t_f]$ into $\tau \in [-1,1]$. This can be done by

\[
t = \frac{t_f}{2}(\tau + 1)
\]

Therefore, each of the constrained time-varying linear optimal control problems (5)-(8) can be reformulated and rewritten in terms of $\tau$ as follows:

For $i \geq 1$

\[
\begin{align*}
\text{Minimize} & \quad J^{[i]} = \frac{t_f}{2} \int_{-1}^{1} (x^{[i]}(\tau)^T Q x^{[i]}(\tau) + u^{[i]}(\tau)^T R u^{[i]}(\tau)) d\tau \\
\text{subject to:} & \quad \text{linearized state equations and initial conditions} \\
& \quad \dot{x}^{[i]} = A(x^{[i-1]}(\tau)) x^{[i]}(\tau) + B(x^{[i-1]}(\tau)) u^{[i]}(\tau), \quad x^{[i]}(-1) = x_0 \\
& \quad G(x(1)^{[i]}) = x_f \\
& \quad X_{\text{min}} \leq x(\tau)^{[i]} \leq X_{\text{max}}, \quad U_{\text{min}} \leq u(\tau)^{[i]} \leq U_{\text{max}}
\end{align*}
\]

where $x^{[0]}(\tau) = x_0$.

V. STATE PARAMETERIZATION VIA CHEBYSHEV POLYNOMIALS

To convert each of the constrained time-varying linear optimal control problems (10)-(13) into quadratic programming problem, the state parameterization using Chebyshev polynomials is applied to approximate the problem as follows:
A) System State Equations Approximation:

The state parameterization technique is implemented according to the work of [9], in which the state variables are approximated as

\[ x_j = \frac{a_j^{(j)}}{2} + \sum_{i=1}^{N} a_i^{(j)} T_i(\tau) \quad j = 1, 2, \ldots, n \]  

(14)

Where \( N \) is the length of the Chebyshev series, \( a_i \) are the unknown parameters and \( T_i \) is a first type Chebyshev polynomial of order \( i \). The control variables are obtained from the system state equations (11) as a function of the unknown parameters of the state variables. These control variables can be rewritten in terms of a finite length series of Chebyshev polynomials with unknown parameters \( b_i \) as follows:

\[ u_l = \frac{a_l^{(j)}}{2} + \sum_{i=1}^{N} b_i^{(j)} T_i(\tau) \quad l = 1, 2, \ldots, m \]  

(15)

Where the unknown parameters \( b_i^{(j)} \) are function of unknown parameters \( a_i^{(j)} \) of the state variables.

To obtain the control variables in terms of Chebyshev polynomials, it is necessary to parameterize the derivative of the state variables. This can be done using Chebyshev polynomials properties as follows [19]:

\[ \dot{x}_j(\tau) = \frac{\dot{a}_j}{2} + \sum_{i=1}^{N-1} \dot{a}_i T_i(\tau) \quad j = 1, 2, \ldots, n \]  

(16)

Where

\[
\begin{align*}
\dot{a}_{N-1} &= 2N a_N \\
\dot{a}_{N-2} &= 2(N - 1) a_{N-1} \\
\dot{a}_{r-1} &= \dot{a}_{r-2} + 2r a_r, \quad r = 1, 2, \ldots, N - 2
\end{align*}
\]

(17)

B) Time-Varying Matrices \( A(x^{[i]-1}(\tau)) \) and \( B(x^{[i]-1}(\tau)) \) Approximation:

The system state equations (11) shows that the two matrices \( A(x^{[i]-1}(\tau)) \) and \( B(x^{[i]-1}(\tau)) \) are a function of \( \tau \), therefore it is necessary to express every \( \tau \) dependant element in both matrices in terms of a Chebyshev series of known parameters. To this end, let \( A_{jl}(\tau) = g(x^{[i]-1}(\tau), \tau) \) be the \( (j,l) \) element of the matrix \( A(x^{[i]-1}(\tau)) \) where \( x^{[i]-1}(\tau) \) is the nominal trajectory of the previous iteration. Then the term \( A_{jl}(\tau) \) can be expressed in terms of a Chebyshev polynomials of known parameters of the form [19]:

\[ A_{jl}(\tau) = \frac{a_{jl}}{2} + \sum_{i=1}^{M} a_i T_i(\tau) \]  

(18)

C) Initial and Terminal State Constraints Approximation:

Using the Chebyshev polynomials initial value property [19], the initial condition vector is approximated as follows

\[ \frac{a_0}{2} - a_1^{(j)} + a_2^{(j)} - a_3^{(j)} + \cdots + (-1)^N a_N^{(j)} - x_0 = 0, \quad j = 1, 2, \ldots, n \]  

(19)

and \( \theta_i = \frac{2i+1}{2K} \pi, i = 1, 2, \ldots, K \) and \( K > N \). The same approximation can be done to the matrix \( B(x^{[i]-1}(\tau)) \).

D) Control and State Saturation Constraints Approximation:

To deal with the saturation constraints on the state and control variables we add a slack variable to the inequality constraint to convert them into equality constraints. This method was used in [7]. However, using this method would result in two drawbacks: The first is adding a slack variable would convert the linear problem into a nonlinear one, while the second drawback is the increase in the number of the unknown parameters.

Another method to deal with the saturation constraints [20, 21] is to discretize the time interval \( \tau \in [-1,1] \) with \( r + 1 \) discrete points, and satisfy the constraints at each point. By this, every continuous constraint is replaced by \( r + 1 \) finite dimension constraints. In this work we will apply this approach.

The time interval \( \tau \in [-1,1] \) is discretized as follows

\[ -1 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_r = 1 \]  

(22)

Therefore, each of the continuous control saturation constraints is replaced by \( r + 1 \) finite dimension inequality constraints. The control saturation constraints are given by

\[ \frac{a_0}{2} + \sum_{i=0}^{N} b_i^{(j)} T_i(\tau) \leq U_{max} \]  

(23)
where $s = 0, 1, 2, ..., r$.

E) Performance Index Approximation:

The last step is to approximate the performance index $J$ in (10). The state and the control variables approximation in (14) and (15) can be rewritten in matrix form as

$$x^{[1]} = S_1 T$$
$$u^{[1]} = S_2 T$$

(27)

Substituting (27) into (10), gives

$$J^{[1]} = \int f_{1}^{[1]}(T^T S_1^T Q S_1 T + S_1^T T R S_1 T) d\tau$$

(28)

Where $J^{[1]}$ is the approximated value of $J^{[1]}$ at iteration $i$. By letting $M = S_1^T Q S_1$ and $P = S_1^T R S_1$ and noting that both matrices $M$ and $P$ are symmetrical, Judd [9] derived an explicit formula for the approximated performance index, therefore $J^{[1]}$ can be expressed as

$$J^{[1]} = \int \sum_{i=1}^{N} \frac{1}{2} \left( \tilde{p}_{i, i+k} + m_{i, i+k} \right) \left( \frac{-2}{2i-2+k^2} \right) d\tau$$

(29)

Where

$$\tilde{p}_{i, i+k} = \begin{cases} p_{i, i+k} & k \neq 0 \\ p_{ii} & k = 0 \end{cases}$$

(30)

$$m_{i, i+k} = \begin{cases} m_{i, i+k} & k \neq 0 \\ m_{ii} & k = 0 \end{cases}$$

(31)

VI. CONTAINER CRANE PROBLEM

In this section, we will apply the proposed method of this research to a practical and complex problem; the container crane. It is desired to transfer containers at the port of Kobe [22] from a ship to a cargo truck. For safety reasons, the objective is to minimize the swing during and at the end of the transfer operation.

Without going into the complex modeling aspects of this problem, which can be found in details in [22], this problem can be state as follows:

Find an optimal controller $u^*(t)$ that minimizes the following performance index

$$J = \frac{1}{2} \int_{0}^{T} (x_2^2 + x_3^2) dt$$

(37)

subject to the following state equations

$$\dot{x}_1 = x_4$$
$$\dot{x}_2 = x_8$$
$$\dot{x}_3 = x_6$$
$$\dot{x}_4 = u_1 + 17,265,6 x_1$$
$$\dot{x}_5 = u_2$$
$$\dot{x}_6 = -\frac{1}{x_2} (u_1 + 27,075,6 x_3 + 2x_5 x_6)$$

(38)-(43)

where

$$X(0) = [0.22, 0.0, -1.0]^T$$

(44)
and

\[ |u_1(t)| \leq 2.83374 \quad \forall t \in [0, 9] \]  
\[ -0.80865 \leq u_2(t) \leq 0.71265 \quad \forall t \in [0, 9] \]

with continuous state inequality constraints

\[ |x_4(t)| \leq 2.5 \quad \forall t \in [0, 9] \]  
\[ |x_5(t)| \leq 1 \quad \forall t \in [0, 9] \]

Applying the iterative technique of section III and changing the time into \( \tau \), we get

For \( i \geq 1 \)

**Minimize**

\[ J[i] = \frac{9}{4} \int_{-1}^{1} (x_3^n)^2 + (x_6^n)^2 d\tau \]  

subject to the following state equations

\[ x_1^n = \frac{9}{2} x_4^n \]  
\[ x_2^n = \frac{9}{2} x_5^n \]  
\[ x_3^n = -\frac{9}{2} x_6^n \]  
\[ x_4^n = \frac{9}{2} (u_1^n - 17.2656 x_2^n) \]  
\[ x_5^n = \frac{9}{2} u_2^n \]

\[ x_6^n = \frac{9}{2} \left( x_4^n - 1 \right) u_1^n + 27.0756 \frac{x_3^n}{x_4^n} + 2 \frac{x_2^n}{x_4^n} \]  

where

\[ x_6^n(-1) = [0.22, 0.0, -1.0]^T \]  
\[ x_6^n(1) = [10.14, 0.25, 0.0]^T \]

and

\[ -2.83374 \leq u_1^n(\tau) \leq 2.83374 \quad \forall \tau \in [-1, 1] \]

This problem was treated by Sakawa and Shindo [22], but no optimal value was reported. Goh and Teo [23] used a piecewise constant functions to parameterize the control variables and \( J \) was found to be 0.005361. They also used a piecewise linear functions to parameterize the control variables and found \( J = 0.005412 \). Jaddu [11] solved this problem using the second method of quasilinearization and state parameterization using Chebyshev polynomials, and \( J \) was found to be 0.00562 after three iterations.

Using the proposed method of this paper, the state variables \( x_1, x_2, x_3 \) were approximated by 9th order Chebyshev series with unknown parameters. The remaining state variables \( x_4, x_5, x_6 \) and control variables \( u_1, u_2 \) are obtained using the state equations (51)-(55). All state equations are directly satisfied except the last equation which will be replaced by \( N + 1 \) equality constraints.

Table 1 illustrates the approximate value of the performance index, while figures 1-8 show the optimal state trajectories and the control variables using the proposed method.

<table>
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<th>Iteration</th>
<th>( J[i] )</th>
</tr>
</thead>
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<tr>
<td>0</td>
<td>0.005200012</td>
</tr>
<tr>
<td>1</td>
<td>0.005644803</td>
</tr>
<tr>
<td>2</td>
<td>0.005644802</td>
</tr>
<tr>
<td>3</td>
<td>0.005644800</td>
</tr>
<tr>
<td>4</td>
<td>0.005644799</td>
</tr>
</tbody>
</table>

Table 1 Approximate Optimal values for the crane problem

Figure 1: \( x_2(t) \) optimal trajectory

[Image of Figure 1]
Figure 2 $x_2(t)$ optimal trajectory

Figure 3 $x_3(t)$ optimal trajectory

Figure 4 $x_4(t)$ optimal trajectory

Figure 5 $x_5(t)$ optimal trajectory

Figure 6 $x_6(t)$ optimal trajectory
VII. CONCLUSION

A numerical method for solving the constrained nonlinear optimal control problems is presented. This method is based using an iterative technique and the state parameterization to replace the problem by sequence of quadratic programming problems. The method is applied on the container crane optimal control problem and the simulation results show that the method give comparable results compared with other methods reported in the literature.

REFERENCES


