Some Fixed Point Theorems for Convex Metric Space taking Random Operator

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Abstract-- In the present paper, some common fixed point theorems are obtained for random operators in convex matrix spaces. We consider the compatible mappings for this purpose. The obtained results include some well known results for special cases.

Keywords-- Convex metric space, Common fixed point, Random operators

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I. INTRODUCTION

Probabilistic functional analysis has emerged as one of the important mathematical disciplines in view of its role in analyzing probabilistic models in the applied sciences. The study of fixed points of random operators forms a central topic in this area. The Prague school of probabilistic initiated its study in the 1950. However, the research in this area flourished after the publication of the survey article of Bharucha-Reid [4]. Since then many interesting random fixed point results and several applications have appeared in the literature; for example the work of Beg and Shahazad [2, 3], Lin [10], O'Regan [11].

In recent years, the study of random fixed points have attracted much attention some of the recent literatures in random fixed points may be noted in [1,2,3,5,12,15]. In particular, random iteration schemes leading to random fixed point of random operators have been discussed in [5,6,7]. Jungck introduced the concept of compatible mappings on metric spaces, as a generalization of weakly commuting mappings, which have been a useful tool for obtaining more comprehensive fixed point theorems. On the other hand, since Takahashi ([15]) introduced a notion of convex metric spaces, many authors have discussed the existence of fixed point and the convergence of iterative processes for nonexpansive mappings in this kind of spaces. The purpose of this paper is to give some fixed point theorems for compatible mappings in convex metric spaces for random operator. Our main results are generalizations of some known results of Choudhary [5, 6, 7], Nan-Jing Huang and Hong Xu Li [12].

II. PRELIMINARIES

Definition 2.1. The pair \((A, B)\) of self-mappings of a metric space \((X, d)\) is said to be compatible on \(X\) if whenever \(\{x_n\}\) is a sequence in \(X\) such that \(Ax_n, Bx_n \to t \in X\), then \(d(BAx_n, ABx_n) \to 0\).

Definition 2.2. Let \((X, d)\) be a metric space and \(J = [0,1]\). A mapping \(W: X \times X \times J \to X\) is called a convex structure on \(X\) if for each \((x, y, \lambda) \in X \times X \times J\) and \(u \in X\),

\[
d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).
\]

A metric space \(X\) together with a convex structure \(W\) is called a convex metric space.

Definition 2.3. A nonempty subset \(K\) of a convex metric space \((X, d)\) with a convex structure \(W\) is said to be convex if for all \((x, y, \lambda) \in K \times K \times J\), \(W(x, y, \lambda) \in K\).

Definition 2.4. Let \(K\) be a convex subset of a convex metric space \((X, d)\) with a convex structure \(W\). A mapping \(I: K \to K\) is said to be \(W\)-affine if for all \((x, y, \lambda) \in K \times K \times J\),

\[
W(x, y, \lambda) = W(Ix, Iy, \lambda).
\]

Throughout this paper, \((\Omega, \Sigma)\) denotes a measurable space, \(C\) is non empty subset of \(K\).

Definition 2.5 Measurable function: A function \(f: \Omega \to C\) is said to be measurable if \(f^{-1}(B \cap C) \in \Sigma\) for every Borel subset \(B\) of \(X\).

Definition 2.6 Random operator: A function \(f: \Omega \times C \to C\) is said to be random operator, if \(F(., X): \Omega \times C \to C\) is measurable for every \(X \in C\).

Definition 2.7 Continuous Random operator: A random operator \(f: \Omega \times C \to C\) is said to be continuous if for fixed \(t \in \Omega\), \(f(t, .): C \to C\) is continuous.

Definition 2.8. Random fixed point: A measurable function \(g: \Omega \to C\) is said to be random fixed point of the random operator \(f: \Omega \times C \to C\), if \(f(t, g(t)) = g(t), \forall t \in \Omega\).
Definition 2.9: Let $(X, d)$ be a metric space and $\left(\Omega, \Sigma\right)$ is a measurable space, $J = [0, 1]$. A mapping $W: X \times X \times J \to X$, is called a convex structure on $X$ for random operator if for each

\[(x(t), y(t), \delta) \in X \times X \times J\] and $u(t) \in X$

\[d(u(t), W(t, (x(t), y(t), \delta))) \leq \delta d(u(t), x(t)) + (1 - \delta)d(u(t), y(t))\]

A metric space $X$ together with a convex structure $w$ and random operator is called a convex random metric space

Definition 2.10: A nonempty subset $K$ of a convex random metric space $(X, d)$ with a convex structure $w$ is said to be convex if for all $(x(t), y(t), \delta) \in K \times K \times J$, $w[t, (x(t), y(t), \delta)] \in K$

**Theorem 3.1.** Let the pair $(F, R)$ be compatible on $K$ such that for all $x, y \in K$,

\[d\left(F(t, x(t)), F(t, y(t))\right) \leq \alpha d\left(R(t, x(t)), R(t, y(t))\right) + \beta \max\left\{d\left(R(t, x(t)), F(t, x(t))\right), d\left(R(t, y(t)), F(t, y(t))\right)\right\} + \gamma d\left(R(t, y(t)), F(t, y(t))\right)\]

\[c_{\max}\left\{d\left(R(t, y(t)), F(t, y(t))\right), d\left(R(t, x(t)), F(t, x(t))\right)\right\}\]

**Proof.** It is clear that $c < 2b(1 - b)/(2 + b)$ implies $a > 0, b > 0$, since $a + b + c = 1$ and $a, b, c$ are nonnegative real numbers.

Let $x = x_0$ be an arbitrary point in $K$ and choose point $x_1, x_2$ and $x_3$ in $K$ such that

\[R(t, x_1(t)) = F(t, x(t)), R(t, x_2(t)) = F(t, x_1(t)), R(t, x_3(t)) = F(t, x_2(t))\]

This can be done since $F(K) \subset R(K)$. For $k = 1, 2, 3$, by (3.1), (3.2) and the triangle inequality, we have

\[d\left(F(t, x_k(t)), R(t, x_{k-1}(t))\right) = d\left(F(t, x_k(t)), F(t, x_{k-1}(t))\right)\]

\[\leq \alpha d\left(R(t, x_k(t)), R(t, x_{k-1}(t))\right) + \beta \max\left\{d\left(R(t, x_k(t)), F(t, x_k(t))\right), d\left(R(t, x_{k-1}(t)), F(t, x_{k-1}(t))\right)\right\}\]

\[+ \gamma \max\left\{d\left(R(t, x_k(t)), F(t, x_k(t))\right), d\left(R(t, x_{k-1}(t)), F(t, x_{k-1}(t))\right)\right\}\]

\[\leq \alpha d\left(R(t, x_k(t)), F(t, x_k(t))\right) + \beta \max\left\{d\left(R(t, x_k(t)), F(t, x_k(t))\right), d\left(R(t, x_{k-1}(t)), F(t, x_{k-1}(t))\right)\right\}\]
If \( d \left( F(t, x_k(t)), R(t, x_k(t)) \right) > d \left( R(t, x_{k-1}(t)), F(t, x_{k-1}(t)) \right) \), it follows from the above inequalities that
\[
d \left( F(t, x_k(t)), R(t, x_k(t)) \right) < (a + b + c)
\]
\[
d \left( R(t, x_k(t)), F(t, x_k(t)) \right) = d \left( F(t, x_k(t)), R(t, x_k(t)) \right).
\]
Which is a contradiction and therefore
\[
d \left( R(t, x_k(t)), F(t, x_k(t)) \right) \leq d \left( R(t, x_{k-1}(t)), F(t, x_{k-1}(t)) \right), k = 1, 2, 3 \tag{3.3}
\]
From (3.1)-(3.3), we get
\[
d \left( R(t, x_1(t)), F(t, x_2(t)) \right) = d \left( F(t, x(t)), F(t, x_2(t)) \right)
\]
\[
\leq a d \left( R(t, x(t)), R(t, x_2(t)) \right) + b \max \left\{ d \left( R(t, x(t)), F(t, x(t)) \right), d \left( R(t, x_2(t)), F(t, x(t)) \right) \right\}
\]
\[
+ c \max \left\{ d \left( R(t, x_1(t)), R(t, x_2(t)) \right), d \left( R(t, x(t)), F(t, x(t)) \right), d \left( R(t, x_2(t)), F(t, x(t)) \right) \right\}
\]
\[
\leq a \left\{ d \left( R(t, x(t)), F(t, x(t)) \right) + d \left( R(t, x_1(t)), F(t, x_1(t)) \right) \right\} + b \left( R(t, x(t)), F(t, x(t)) \right)
\]
\[
+ c \max \left\{ d \left( R(t, x(t)), F(t, x(t)) \right), d \left( R(t, x_1(t)), F(t, x_1(t)) \right), d \left( R(t, x_2(t)), F(t, x(t)) \right) \right\}
\]
\[
\leq (2a + b + 2c) d \left( R(t, x(t)), F(t, x(t)) \right)
\]
\[
\leq (2 + b) d \left( R(t, x(t)), F(t, x(t)) \right)
\]

Let \( z = W \left( x_2, x_3, \frac{1}{2} \right) \). Then we know that \( z \in K \) and
\[
R(t, z(t)) = W \left( R(t, x_2(t)), R(t, x_3(t)), \frac{1}{2} \right) = W \left( F(t, x_1(t)), F(t, x_2(t)), \frac{1}{2} \right).
\]
By (3.2)-(3.4), we obtain
\[
d \left( R(t, z(t)), R(t, x_1(t)) \right) = d \left( R(t, x_1(t)), W \left( F(t, x_1(t)), F(t, x_2(t)), \frac{1}{2} \right) \right)
\]
\[
\leq \frac{1}{2} d \left( R(t, x_1(t)), F(t, x_1(t)) \right) + \frac{1}{2} d \left( R(t, x_1(t)), F(t, x_2(t)) \right)
\]
\[\frac{3 - b}{2} d \left( R(t, x(t)), F(t, x(t)) \right). \quad (3.5)\]

\[d \left( R(t, z(t)), R(t, x_2(t)) \right) = d \left( R(t, x_2(t)), W \left( F(t, x_1(t)), F(t, x_2(t)) \right) \right)^{\frac{1}{2}}\]

\[\leq \frac{1}{2} d \left( R(t, x_2(t)), F(t, x_1(t)) \right) + \frac{1}{2} d \left( R(t, x_2(t)), F(t, x_2(t)) \right)\]

\[\leq \frac{1}{2} d \left( R(t, x(t)), F(t, x(t)) \right) \quad (3.6)\]

\[d \left( R(t, z(t)), R(t, x_3(t)) \right) = d \left( R(t, x_3(t)), W \left( F(t, x_1(t)), F(t, x_2(t)) \right) \right)^{\frac{1}{2}}\]

\[\leq \frac{1}{2} d \left( R(t, x_3(t)), F(t, x_1(t)) \right) + \frac{1}{2} d \left( R(t, x_3(t)), F(t, x_2(t)) \right)\]

\[\leq \frac{1}{2} d \left( R(t, x(t)), F(t, x(t)) \right). \quad (3.7)\]

It follows from (3.1)-(3.7) that

\[d \left( R(t, z(t)), F(t, x(t)) \right)\]

\[= d \left( F(t, z(t)), W \left( F(t, x_1(t)), F(t, x_2(t)) \right) \right)^{\frac{1}{2}}\]

\[\leq \frac{1}{2} d \left( F(t, z(t)), F(t, x_1(t)) \right) + \frac{1}{2} d \left( F(t, z(t)), F(t, x_2(t)) \right)\]

\[\leq \frac{1}{2} a d \left( R(t, z(t)), R(t, x_1(t)) \right) + \frac{1}{2} b \max \left\{ d \left( R(t, z(t)), F(t, x(t)) \right) \right\}\]

\[+ \frac{1}{2} c \max \left\{ d \left( R(t, z(t)), R(t, x_1(t)) \right), d \left( R(t, z(t)), F(t, x(t)) \right) \right\}\]

\[\leq \frac{a(3 - b)}{4} d \left( R(t, x(t)), F(t, x(t)) \right)\]

\[+ \frac{1}{2} b \max \left\{ d \left( R(t, z(t)), F(t, x(t)) \right), d \left( R(t, x(t)), F(t, x(t)) \right) \right\}\]

\[+ \frac{1}{2} c \max \left\{ d \left( R(t, z(t)), F(t, x(t)) \right), d \left( R(t, x(t)), F(t, x(t)) \right), d \left( R(t, x_2(t)), F(t, x(t)) \right) \right\}\]

\[\leq \frac{4 - b}{4} d \left( R(t, x(t)), F(t, x(t)) \right) + \frac{1}{2} d \left( R(t, z(t)), F(t, z(t)) \right)\]
\[ + \frac{1}{4} a \left( R(t, x(t)), F(t, x(t)) \right) + \frac{1}{2} b \max \left\{ d \left( R(t, z(t)), F(t, z(t)) \right), d \left( R(t, x(t)), F(t, x(t)) \right) \right\} \]

\[ + \frac{1}{2} c \max \left\{ d \left( R(t, x(t)), F(t, x(t)) \right), d \left( R(t, z(t)), F(t, z(t)) \right) \right\} \]

If \( d \left( R(t, z(t)), F(t, z(t)) \right) > d \left( R(t, x(t)), F(t, x(t)) \right) \), then the above inequalities imply that

\[ d \left( R(t, z(t)), F(t, z(t)) \right) \leq \left( \frac{a(4-b)}{4} + b + \frac{c(10-b)}{8} \right) d \left( R(t, z(t)), F(t, z(t)) \right) \]

\[ = \left( 1 - \frac{1}{8} (2ab + bc - 2c) \right) d \left( R(t, z(t)), F(t, z(t)) \right). \]

This is a contradiction, since \( \frac{2b(1-b)}{2+b} > c \) and \( a + b + c = 1 \) imply \( 2ab + bc - 2c > 0 \). Hence, we have

\[ d \left( R(t, z(t)), F(t, z(t)) \right) \leq \left( 1 - \frac{1}{8} (2ab + bc - 2c) \right) d \left( R(t, x(t)), F(t, x(t)) \right). \quad (3.8) \]

Letting \( \lambda = 1 - \frac{(2ab+bc-2c)}{8} \), we know that \( 0 < \lambda < 1 \). Since \( x \) is an arbitrary point in \( K \), it follows from (3.8) that there exists a sequence \( \{z_n\} \) in \( K \) such that

\[ d \left( R(t, z_0(t)), F(t, z_0(t)) \right) \leq \lambda d \left( R(t, x_0(t)), F(t, x_0(t)) \right) \]

\[ d \left( R(t, z_1(t)), F(t, z_1(t)) \right) \leq \lambda d \left( R(t, z_0(t)), F(t, z_0(t)) \right). \]

\[ : \]

\[ d \left( R(t, z_n(t)), F(t, z_n(t)) \right) \leq \lambda d \left( R(t, z_{n-1}(t)), F(t, z_{n-1}(t)) \right) \]

Which yield that \( d \left( R(t, z_n(t)), F(t, z_n(t)) \right) \leq \lambda^{n+1} d \left( R(t, z_0(t)), F(t, z_0(t)) \right) \) and so we have

\[ d \left( R(t, z_n(t)), F(t, z_n(t)) \right) \to 0. \quad (3.9) \]

Setting

\[ K_n = \left\{ x \in K : d \left( F(t, x(t)), R(t, x(t)) \right) \leq \frac{1}{n} \right\}, n = 1, 2, \ldots, \]

Then (3.9) implies that \( K_n \neq \emptyset, n = 1, 2, \ldots, \) and \( K_1 \supset K_2 \supset \cdots \). Obviously,

\[ TK_n \neq \emptyset \text{ and } TK_n \supset TK_{n+1}, n = 1, 2, \ldots. \]

For any \( x, y \in K_n \), it follows from (3.1) that

\[ d \left( F(t, x(t)), F(t, y(t)) \right) \leq a \left( R(t, x(t)), F(t, x(t)) \right) \]

\[ \leq a d \left( R(t, x(t)), F(t, y(t)) \right) \]

\[ + b \max \left\{ d \left( R(t, x(t)), F(t, x(t)) \right), d \left( R(t, y(t)), F(t, y(t)) \right) \right\} \]
\[ d \left( R(t,x(t)), R(t,y(t)) \right) \leq \max \left\{ \frac{1}{2} d \left( R(t,x(t)), F(t,y(t)) \right), \frac{1}{2} d \left( R(t,y(t)), F(t,x(t)) \right) \right\} + c \max \left\{ d \left( R(t,x(t)), F(t,y(t)) \right), d \left( R(t,y(t)), F(t,x(t)) \right) \right\} \]

\[ \leq a \left( \frac{1}{n^2} + d \left( F(t,x(t)), F(t,y(t)) \right) \right) + b \frac{1}{n} + c \left( \frac{1}{n^2} + d \left( F(t,x(t)), F(t,y(t)) \right) \right) \]

\[ = \frac{1}{n} \left( 2a + b + 2c \right) + (a + c) d \left( F(t,x(t)), F(t,y(t)) \right) \]

Which yields

\[ d \left( F(t,x(t)), F(t,y(t)) \right) \leq \frac{1}{nb} (2a + b + 2c). \]

Therefore, we have

\[ \lim_{n \to \infty} \text{diam} \left( \overline{T K_n} \right) = \lim_{n \to \infty} \text{diam} \left( TK_n \right) = 0. \]

Where \text{diam}(TK_n) denotes the diameter of \overline{TK_n}. By the Cantor’s theorem, there exists a point \( u \in K \) such that \( \{ u \} = \bigcap_{n=1}^{\infty} TK_n \). Since \( u \in K \), for each \( n = 1, 2, \ldots \), there exists a point \( x_n \in K_n \) such that \( d \left( u, F(t,x_n(t)) \right) < \frac{1}{n^2} \), and so \( F(t,x_n(t)) \to u \).

Further, since \( x_n \in F_n \), we have

\[ d \left( F(t,x_n(t)), R(t,x_n(t)) \right) < \frac{1}{n^2}, \quad \text{and} \quad R(t,x_n(t)) \to u. \]

Since \( R \) is continuous and the pair \( (F,R) \) is compatible, we can induce easily that

\[ RF(t,x_n(t)), RR(t,x_n(t)), FR(t,x_n(t)) \to R(t,u(t)). \]

Now, (3.1) yields that

\[ d \left( F(t,u(t)), R(t,u(t)) \right) \leq d \left( F(t,u(t)), FR(t,x_n(t)) \right) + d \left( FR(t,x_n(t)), R(t,u(t)) \right) \]

\[ \leq a \max \left\{ \frac{1}{n^2} d \left( R(t,u(t)), F(t,u(t)) \right), \frac{1}{n^2} d \left( R(t,u(t)), F(t,u(t)) \right) \right\} + b \max \left\{ d \left( R(t,u(t)), F(t,u(t)) \right), d \left( R(t,u(t)), F(t,u(t)) \right) \right\} \]

\[ + c \max \left\{ d \left( R(t,u(t)), F(t,u(t)) \right), d \left( R(t,u(t)), F(t,x_n(t)) \right) \right\} + d \left( FR(t,x_n(t)), R(t,u(t)) \right). \]

Letting \( n \to \infty \), we have

\[ d \left( F(t,u(t)), R(t,u(t)) \right) \leq (b + c) d \left( F(t,u(t)), R(t,u(t)) \right). \]

Which leads to \( d \left( R(t,u(t)), F(t,u(t)) \right) = 0 \), and so \( R(t,u(t)) = F(t,u(t)) \). Hence \( RR(t,u(t)) = RF(t,u(t)) = FR(t,u(t)) = FF(t,u(t)) \), since the pair \( (F,R) \) is compatible. Using (3.1), we get

\[ d \left( FR(t,u(t)), F(t,u(t)) \right) \leq d \left( FR(t,u(t)), R(t,u(t)) \right) \]

\[ + b \max \left\{ d \left( FR(t,u(t)), F(t,u(t)) \right), d \left( R(t,u(t)), F(t,u(t)) \right) \right\} \]
This implies that \( d(FF(t,u(t)), F(t,u(t))) = 0 \), since \( b > 0 \) and \( a + b + c = 1 \). Therefore, \( FF(t,u(t)) = F(t,u(t)) \).

\[
\begin{align*}
  d(z,z_1) &= d(F(t,z(t)), F(t,z_1(t))) \\
  &\leq a \ d(R(t,z(t)), R(t,z_1(t))) + b \ \max \left\{ \frac{d(R(t,z(t)), F(t,z(t)))}{d(R(t,z(t)), F(t,z(t)))}, \frac{d(R(t,z_1(t)), F(t,z_1(t)))}{d(R(t,z_1(t)), F(t,z_1(t)))} \right\} \\
  &+ c \ \max \left\{ \frac{d(R(t,z(t)), F(t,z(t)))}{d(R(t,z(t)), F(t,z(t)))}, \frac{d(R(t,z_1(t)), F(t,z_1(t)))}{d(R(t,z_1(t)), F(t,z_1(t)))} \right\} \\
  &= (a + c)d(z,z_1).
\end{align*}
\]

Which is a contradiction since \( a + c = 1 - b < 1 \). Hence, \( z \) is the unique common fixed point of \( F \) and \( R \).

Now, we prove that \( F \) is continuous at \( z \). Let \( \{z_n\} \subset K \) and \( z_n \to z \). Since \( R \) is continuous, \( R(t,z_n(t)) \to R(t,z(t)) \). By (3.1), we have

\[
\begin{align*}
  d(F(t,z_n(t)), F(t,z(t))) &\leq a \ d(R(t,z_n(t)), R(t,z(t))) + b \ \max \left\{ \frac{d(R(t,z_n(t)), F(t,z(t)))}{d(R(t,z_n(t)), F(t,z(t)))}, \frac{d(R(t,z(t)), F(t,z(t)))}{d(R(t,z(t)), F(t,z(t)))} \right\} \\
  &+ c \ \max \left\{ \frac{d(R(t,z_n(t)), F(t,z(t)))}{d(R(t,z_n(t)), F(t,z(t)))}, \frac{d(R(t,z(t)), F(t,z(t)))}{d(R(t,z(t)), F(t,z(t)))} \right\} \\
  \end{align*}
\]

This leads to

\[
\lim_{n \to \infty} \sup d(F(t,z_n(t)), F(t,z(t))) \leq (b + c) \lim_{n \to \infty} \sup d(F(t,z_n(t)), F(t,z(t)))
\]

And so \( \lim_{n \to \infty} d(F(t,z_n(t)), F(t,z(t))) = 0 \) since \( b + c = 1 - a < 1 \). Therefore, \( F \) is continuous at \( z \). This completes the proof.
\[ \begin{aligned}
&d \left( F(t, x(t)), F(t, y(t)) \right) \\
&\quad \leq a \left( d \left( R(t, x(t)), R(t, y(t)) \right) \right) \\
\quad &+ (1 - a) \max \left\{ d \left( R(t, x(t)), F(t, x(t)) \right), d \left( R(t, y(t)), F(t, y(t)) \right) \right\}
\end{aligned} \]

For all \( x, y \in K \), where \( 0 < a < 1 \) is a constant. If \( F(K) \subseteq R(K) \) and \( R \) is \( W \)-affine and continuous, then \( F \) and \( R \) have a unique common random fixed point \( z \in K \) and \( F \) is continuous at \( z \).

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