Coincidence and Common Fixed Point Theorem in Fuzzy Metric Space

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Abstract--Singh and Jain introduced the concept of semicompatibility in fuzzy metric space and it has been applied to prove results on existence of unique common fixed point of self mappings satisfying an implicit relation. In this paper we improve the results of Singh and Jain without using semicompatibility.

I. INTRODUCTION

Using the concept of induced fuzzy topology, Michael D. Weiss [11] proved a generalization of fuzzy sets of the well known Schauder-Tychonoff fixed point theorem. Heilpern [4] discussed the concept of fuzzy mappings (mapping from an arbitrary set to one subfamily of fuzzy sets in a metric linear space) and proved a generalization of fixed point theorem for point-to-set mappings [4].

Bose and Sahani [1] extended the result of Heilpern for a generalized fuzzy contraction. They also proved a fixed point theorem for a nonexpansive fuzzy mappings on a starshaped subset of a Banach space. Chang [2] obtained a coincidence theorem for fuzzy mappings on a metric space. Many authors ([4], [6] & [7]) obtained common fixed point theorems for weakly commuting maps and R–weakly commuting mappings.

In 2005, Singh and Jain [9] introduced the concept of semi compatibility in fuzzy metric space and it has been applied to prove unique common fixed point theorem of four self mappings on fuzzy metric space. In this paper we improve the results of Singh and Jain [9].

Definition 1.1. A binary operation \(* : [0,1] \times [0,1] \rightarrow [0,1]\) is said to be continuous \(t\)-norm if \(\ast\) satisfies the following conditions:

(i) \(\ast\) is commutative and associative.
(ii) \(\ast\) is continuous.
(iii) \(a \ast 1 = a\) for all \(a \in [0,1]\).

\(a \ast b \leq c \ast d\) whenever \(a \leq c\) and \(b \leq d\) for all \(a,b,c,d \in [0,1]\).

Definition 1.2. (Kramosil and Michalek[8]). The three tuple \((X, M, \ast)\) is said to be fuzzy metric space if \(X\) is an arbitrary set, \(\ast\) is a continuous \(t\)-norm and \(M\) is a fuzzy set in \(X^2 \times [0, \infty)\) satisfying the following conditions for all \(x, y, z\) in \(X\) and \(s, t > 0\):

(i) \(M(x, y, 0) = 0\),
(ii) \(M(x, y, t) = 1\), if and only if \(x = y\),
(iii) \(M(x, y, t) = M(y, x, t)\),
(iv) \(M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s)\),

and (v) \(M(x, y, \cdot) : [0, \infty) \rightarrow [0,1]\) is left continuous.

Definition 1.3 (Grabiec [5]). Let \((X, M, \ast)\) be a fuzzy metric space. A sequence \(\{x_n\}\) in \(X\) is said to be convergent to a point \(x\) in \(X\) if

\[\lim_{n \rightarrow \infty} M(x_n, x, t) = 1 \quad \text{for all } t > 0.\]

Further the sequence \(\{x_n\}\) is said to be Cauchy sequence in \(X\) if

\[\lim_{n \rightarrow \infty} M(x_n, x_n + p, t) = 1 \quad \text{for all } t > 0 \quad \text{and} \quad p > 0.\]

The space \((X, M, \ast)\) is said to be complete if every Cauchy sequence in it converges to a point of it.

In this paper, \((X, M, \ast)\) is considered to be the fuzzy metric space with the condition,

\[\lim_{n \rightarrow \infty} M(x, y, t) = 1 \quad \text{for all } x, y \text{ in } X.\] (1.1)

Lemma 1.2 (Cho [3]). Let \(\{y_n\}\) be a sequence in fuzzy metric space \((X, M, \ast)\) with the condition (1.1). Suppose there exists a number \(k \in (0,1)\) such that,

\[M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t), \quad \text{for all } t > 0.\]

Then \(\{y_n\}\) is a Cauchy sequence.
Definition 1.4. Let $f$ and $g$ be mappings from a fuzzy metric space $(X, M, \cdot)$ to itself. The mappings are said to be weak compatible if they commute at their coincidence points, that is,

$$fx = gx \implies fgx = gfx.$$  

Definition 1.5. Let $f$ and $g$ be mappings from a fuzzy metric space $(X, M, \cdot)$ to itself. Then the mappings are said to be compatible if,

$$\lim_{n \to \infty} M(fgy_n, gfy_n, t) = 1, \quad \forall \ t > 0,$$

whenever $\{y_n\}$ is a sequence in $X$ such that,

$$\lim_{n \to \infty} fy_n = \lim_{n \to \infty} gy_n = x \in X.$$

Proposition 1.1 [10]. Self mappings $f$ and $g$ of a metric space $(X, M, \cdot)$ are compatible, then they are weak compatible.

Definition 1.6 (Singh And Jain [9]). Let $f$ and $g$ be mappings from a fuzzy metric space $(X, M, \cdot)$ to itself. Then the mappings are said to be semicompatible if

$$\lim_{n \to \infty} M(fgy_n, gx, t) = 1, \quad \forall \ t > 0,$$

whenever $\{y_n\}$ is a sequence in $X$ such that,

$$\lim_{n \to \infty} fy_n = \lim_{n \to \infty} gy_n = x \in X.$$

Singh and Jain [9] proved that if the mappings $f$ and $g$ are semicompatible, then they are weak compatible without the converse being true.

II. MAIN RESULTS

Theorem 2.1. Let $(X, M, \cdot)$ be a fuzzy metric space and $Y$ is an arbitrary set. Suppose $k \in (0,1)$ and $f, g, h: Y \to X$ are mappings such that,

(i) $M(fx, gy, kt) \geq \min\{M(hx, hy, t), M(fx, hx, t)\}$ $\forall \ x, y \in Y$ and $t > 0$,

(ii) $f(Y) \cup g(Y) \subset h(Y)$,

and (iii) one of $f(Y), g(Y), h(Y)$ is complete.

Then $f, g$ and $h$ have coincidence point.

Proof. For $p_0 \in Y$ there exist $p_1, p_2 \in Y$ such that $fp_0 = hp_1, gp_1 = hp_2$ (because $f(Y) \cup g(Y) \subset h(Y)$).

Inductively we can construct a sequence $\{p_n\}$ such that $fp_{2n} = hp_{2n+1}, gp_{2n+1} = hp_{2n+2}$. Putting $x = p_{2n}$ and $y = p_{2n+1}$ in (i), we have

$$M(fp_{2n}, gp_{2n+1}, kt) \geq \min\{M(hp_{2n}, hp_{2n+1}, t), M(fp_{2n}, hp_{2n}, t)\}$$

i.e. $M(hp_{2n+1}, gp_{2n+2}, kt) \geq M(hp_{2n+1}, gp_{2n+2}, t)$.

So by Lemma 1.1 $\{hp_n\}$ is a Cauchy sequence. Suppose $h(Y)$ is complete. Then $\{hp_n\} \to p \in h(Y)$.

Also then there exists $u \in Y$ such that $hu = p$. Putting $x = u, y = p_{2n+1}$ in (i), we have,

$$M(fu, gp_{2n+1}, kt) \geq \min\{M(hu, hp_{2n+1}, t), M(fu, hu, t)\}$$

Therefore in limiting case as $n \to \infty$, $M(fu, p, kt) \geq \min\{M(p, p, t), M(fu, hu, t)\}$

i.e. $M(fu, p, kt) \geq M(fu, p, t)$.

Hence $f(u) = p = h(u)$. Lastly, putting $x = p_{2n+1}, y = u$ in (i), we have,

$$M(fp_{2n+1}, gu, kt) \geq \min\{M(hp_{2n+1}, hu, t), M(fp_{2n+1}, hp_{2n+1}, t)\}$$

i.e. $M(p, gu, kt) \geq M(p, p, t)$.

Therefore $p = gu = f(u) = h(u)$. Hence $u$ is coincidence point of $f, g$ and $h$.

Theorem 2.2. Let $(X, M, \cdot)$ be a fuzzy metric space. Suppose $k \in (0,1)$ and $f, g, h: X \to X$ are mappings such that,


(i) \( M(fx, gy, kt) \geq \min\{M(hx, hy, t), M(fx, hx, t)\} \quad \forall \ x, y \in X \) and \( t > 0 \),

(ii) \( f(X) \cup g(X) \subseteq h(X) \),

(iii) one of \( f(X), g(X), h(X) \) is complete,

and (iv) \( f \) and \( h \) are coincidently commuting.

Then \( f, g \) and \( h \) have a unique common fixed point.

Proof. In the theorem 2.1 if we take \( Y = X \) then we get \( p = gu = f(u) = h(u) \).

Since \( f \) and \( g \) are coincidently commuting so,

\[
flu = hfu \quad \text{and} \quad fp = hp.
\]

Putting \( x = fu, y = p_{2n+1} \) in (i), we have,

\[
M(fffu, gp_{2n+1}, kt) \geq M(hfu, hp_{2n+1}, t), M(fffu, hfu, t).
\]

Taking \( n \to \infty \), we have, \( M(fffu, p, kt) \geq M(fp, p, t) \).

Therefore \( fp = hp = p \).

Again putting \( x = p_{2n}, y = p \) in (i) and taking limit as \( n \to \infty \) we get,

\[
gp = fp = hp = p. \text{Hence} \ p \text{ is a common fixed point}\ of\ f, g \text{ and } h.
\]

For uniqueness suppose \( p \) and \( q \) are common fixed points of \( f, g \) and \( h \).

Then by putting \( x = p, y = q \) in (i) we get \( p = q \). This proves the uniqueness.

Theorem 2.3 ([Singh And Jain [191]]). Let \( f \) and \( g \) be two self mappings on a complete fuzzy metric space \((X, M, \ast)\) such that for some \( k \in (0,1)\), for all \( x, y \in X \) and \( t > 0 \),

\[
M(fx, gy, kt) \geq \min\{M(x, y, t), M(fx, x, t)\}.
\]

Then \( f \) and \( g \) have a unique common fixed point.

Proof. Taking \( h = I \) (Identity map) in Theorem 2.2 we get the result.

REFERENCES: