Analytical Study of Compact Mapping in Hilbert Space

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Abstract-- This paper presents the study of the representation of compact mappings in Hilbert Space. Here we denote the Scalar Product of two elements (x,y) of a (real or complex) Hilbert Space by (x,y). Here it is proved in this paper that the study of compact mappings in Hilbert Space is a consequence of the spectral theory of compact Symmetric operators.

Keywords-- Hilbert Space, Compact mapping , Spectral theory of compact Symmetric operators, Orthogonal projection, Riesz Representation, Scalar Product.

I. INTRODUCTION

Hall (1) and Kothe (2,3) are the pioneer worker of the present area. In fact, the present work is the extension of work done by Wong, Yau-Chuen (10), Srivastava et al. (4), Srivastava et al. (5), Srivastava et al. (6), Srivastava et al. (7), Kumar et al. (8) and Srivastava et al. (9). In this paper we have studied analytically about compact mapping in Hilbert Space.

Here, we use the following definitions, Notations and Fundamental ideas:

If M and N are subspaces of a Linear space X such that every x ∈ X can be written uniquely as x = y + z where y ∈ M & z ∈ N then the direct sum of M and N can also be written X = M ⊕ N where N is called complimentary subspace of M in X and if M ∩ N = {0}, the decomposition x = y + z is unique.

A given subspace M has many complimentary subspaces and every complimentary subspace of M has the same dimension and the dimension of a complimentary subspace is called co-dimension of M in X, as if X = R² and M is a plane through the origin then any line through the origin that does not lie in M is a complimentary subspace.

If X = M ⊕ N then we define the projection P: X → X of x on to M along N by Px = y, where x = y + z with y ∈ M, z ∈ N which is Linear with ran P = M and ker P = N satisfying P² = P.

This property characterizes projections for which the following definitions and theorems follow:

**Definition 1**: Any projection associated with a direct sum decomposition of a projection on a Linear space X is a linear map P: X → X such that P² = P

**Definition 2**: An orthogonal projection on a Hilbert space H is also a Linear mapping P: H → H satisfying P² = P, <Px,y> = <x, Py> for all x, y ∈ H.

“An orthogonal projection is necessarily bounded.”

**Theorem 1**: Let X be a linear space, then

(i) If P: X → X is a projection then X = ran P ⊕ ker P

(ii) If X = M ⊕ N where M and N are Linear subspaces of X then there is a projection P: X → X with ran P = M and ker P = N.

**Proof:**

For (i) We show that x ∈ ran P if x = Px

If x = Px then clearly x ∈ ran P

If x ∈ ran P then x = Py for some y ∈ X

And since P² = P which follows that Px = P²y = Py = x

If x ∈ ran P ∩ ker P then x = Px & Px = 0

So ran P ∩ ker P = {0}. If x ∈ X then

We have x = Px + (x - Px); where Px ∈ ran P and (x - Px) ∈ ker P.

Since P (x - Px) = Px - P²x = Px - Px = 0

Thus X = ran P ⊕ ker P. ..........................(1.1)

Now for (ii)

We consider if X = M ⊕ N then x ∈ N has unique decomposition x = y + z with y ∈ M & z ∈ N and Px = y defines the required Projection.
In particular, in orthogonal subspaces while using Hilbert Space, let us suppose that $M$ is a closed subspace of Hilbert Space $H$ then by well known property we have $H = M \oplus M^\perp$. We call the projection of $H$ on to $M$ along $M^\perp$ the orthogonal projection of $H$ on to $M$.

If $x = y + z$ and $x_1 = y_1 + z_1$ where $y, y_1 \in M$ and $z, z_1 \in M^\perp$ then by orthogonality of $M$ and $M^\perp \implies \langle Px, x_1 \rangle = \langle y, y_1 + z_1 \rangle = \langle y, y_1 \rangle = \langle y + z, y_1 \rangle = \langle x, Px_1 \rangle$ (1.2)

Which states that an orthogonal projection is self Adjoint. We show the properties (1.1) and (1.2) characterize orthogonal projections with Defn-2.

**Lemma:** If $P$ is a non zero orthogonal projection then $\| P \| = 1$.

**Proof:** If $x \in H$ and $Px \neq 0$ then by Cauchy Schwarz inequality,

$$\| Px \| = \langle Px, Px \rangle = \langle x, P^2x \rangle = \langle x, Px \rangle \leq \| x \| \langle Px \| \| Px \|$$

Therefore $\| P \| \leq 1$. If $P \neq 0$ then there is an $x \in H$ with $Px \neq 0$ and $\| P( Px) \| = \| Px \|$ so that $\| P \| \geq 1$.

Thus, the Orthogonal Projection $P$ and closed subspace $M$ of $H$ such that $ran P = M$ will must obey one one correspondence, then the kernel of Orthogonal Projection is the Orthogonal Complement of $M$.

**Example.1** – The space $L^2 (R)$ is the Orthogonal direct sum of space $M$ of even functions and the space $N$ of odd functions.

The Orthogonal Projection $P$ and $Q$ of $H$ onto $M$ and $N$, respectively are given by

$$Pf(x) = \frac{f(x) + f(-x)}{2} , Qf(x) = \frac{f(x) - f(-x)}{2}$$

Where $I-P = Q$.

**Proposition:** (a) A Linear functional on a Complex Hilbert space $H$ is a Linear map from $H$ to $C$. A Linear functional $\varphi$ is bounded or continuous, if there exists a constant $M$ such that $| \varphi(x) | \leq M \| x \|$ for all $x \in H$.

The norm of bounded linear functional $\varphi$ is

$$\| \varphi \| = \sup | \varphi(x) |$$

$$\| x \| = 1$$

If $y \in H$ then $\varphi_y(x) = \langle y, x \rangle$ is a bounded Linear functional on $H$, with

$$\| \varphi_y \| = \| y \| .$$

(b) If $\varphi$ is a bounded Linear functional on a Hilbert space $H$, then there is a unique vector $y \in H$ such that

$$\varphi(x) = \langle y, x \rangle \quad \text{for all } x \in H$$

**Theorem.2 :** (Riesz representation) If $\varphi$ is a bounded linear functional on a Hilbert space $H$, then there is a unique vector $y \in H$ such that

$$\varphi(x) = \langle y, x \rangle \quad \text{for all } x \in H \quad \text{……..(2.1)}$$

**Proof.** If $\varphi = 0$, then $y = 0$, so we suppose that $\varphi \neq 0$. In that case, ker$\varphi$ is a proper closed subspace of $H$ and, it implies that, there is a nonzero vector $z \in H$ such that $z \perp$ ker$\varphi$. We define a linear map $P : H \rightarrow H$ by

$$Px = \varphi(x)/\varphi(z)z$$

Then $P^2 = P$, so Theorem 1 implies that $H = \text{ran } P \oplus \ker P$. Moreover,

$$\text{ran} P = \{az|a \in C\}, \ker P = \ker \varphi$$

So that ran $P \perp \ker P$. It follows that $P$ is an orthogonal projection, and

$$H = \{az|a \in C\} \oplus \ker \varphi$$

is an orthogonal direct sum. We can therefore write

$$x \in H \text{ as } x = az + n, a \in C \text{ and } n \in \ker \varphi.$$ 

Taking the inner product of this decomposition with $z$, we get

$$a = \langle z, z \rangle/\| z \|^2,$$

and evaluating $\varphi$ on $x = az + n$, we find that

$$\varphi(x) = a \varphi(z).$$

The elimination $a$ from these equations, and a rearrangement of the result

yields $\varphi(x) = \langle y, x \rangle$, where $y = \varphi(z)/\| z \|^2z$.

Thus, every bounded linear functional is given by the inner product with a fixed vector. We have already seen that $\varphi_y(x) = \langle y, x \rangle$ defines a bounded linear functional on $H$ for every $y \in H$. 

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To prove that there is a unique $y$ in $H$ associated with a given linear functional, suppose that $\varphi_y \equiv \varphi_{y_2}$. Then $\varphi_{y_1}(y) = \varphi_{y_2}(y)$. When $y = y_1 = y_2$, which implies that $\Pi y_1 - y_2 \Pi^2 = 0$, so $y_1 = y_2$.

The map $J: H \to H^*$ given by $J_y = \varphi_y$, therefore identifies a Hilbert space $H$ with its dual space $H^*$. The norm of $\varphi_y$ is equal to the norm of $y$, so $j$ is an isometry. In this case of complex Hilbert spaces, $J$ is antilinear, rather than linear, because $\varphi_{xy} = \lambda \varphi_y$. Thus, Hilbert spaces are self-adjoint, meaning that $H$ and $H^*$ are isomorphic as Banach spaces, and anti-isomorphic as Hilbert spaces. Thus Hilbert spaces are special in this respect. This completes the proof of the Theorem 2.

**Proposition:** (c) An important consequence of the Riesz representation theorem is the existence of the adjoint of a bounded linear operator on a Hilbert space. The definition property of the adjoint $A^* \in B(H)$ of an operator $A \in B(H)$ is that
\[
\langle x, Ay \rangle = \langle A^*x, y \rangle \quad \text{for all } x, y \in H \ldots \ldots \ldots \quad (2.2)
\]
The Uniqueness of $A^*$ is obvious. The definition implies that
\[
(A^*)^* = A, \quad (AB)^* = B^*A^*.
\]
To prove that $A^*$ exists, we have to show that for every $x \in H$, there is a vector $z \in H$, depending linearly on $x$ such that

\[
\langle z, y \rangle = \langle x, Ay \rangle \quad \text{for all } x, y \in H \ldots \ldots \ldots \quad (2.3)
\]

For fixed $x$, the map $\varphi_x$ defined by $\varphi_x(y) = \langle x, Ay \rangle$ is a bounded linear functional on $H$, with $\|\varphi_x\| \leq II_{II}$. By the Riesz representation theorem, there is a unique $z \in H$ such that $\varphi_x(y) = \langle z, y \rangle$. This $z$ satisfies (2.3), so we get $A^*x = z$. The linearity of $A^*$ follows from the uniqueness in the Riesz representation theorem and the linearity of the inner product.

Thus, from above definitions, theorems, lemmas, examples, propositions (a), (b), & (C), which shows the proof of the main result as “the representation of compact mappings of Hilbert spaces is a consequence of the spectral theory of compact symmetric operators.

1) Let $H_1, H_2$ be Hilbert spaces, $A \in \mathcal{L}(H_1, H_2)$ compact and not of finite rank. Then, there exists orthonormal systems, $e_n$, $n = 1, 2, \ldots \ldots \ldots$. In $H_1$ and $\{f_n\}$, $n = 1, 2, \ldots \ldots \ldots$ in $H_2$ such that $\infty$

2) $A x = \sum_{n=1}^{\infty} \lambda_n (x, e_n) f_n$, $x \in H_1$ where $\lambda_n > 0$ and $\lambda_n \to 0$.

**Proof:** Since $A$ is Compact, $A^*A$ is Compact too and positive, where $A^*$ denotes the adjoint in the sense of the scalar product. It follows from Spectral theory that there exists an orthonormal sequence of eigen vectors $e_n$, $n = 1, 2, 3, \ldots \ldots$ and eigen values $\lambda_n^2 > 0$, $\lambda_n^2 \to 0$ such that
\[
\sum_{n=1}^{\infty} \lambda_n^2 (x, e_n) e_n.
\]

$A^*A$ is zero on the orthonormal or complement $H$ of the closed subspace spanned by all the $e_n$. But then $A$ is zero too on $H$.

Take $y \in H$ and suppose $A_y \neq 0$.

Then $(A_y, A_x) = (y, A^*A_x) \neq 0$. But this would imply $A^*A_x \neq 0$. Therefore we have a representation
\[
\sum_{n=1}^{\infty} \lambda_n^2 (x, e_n) A e_n.
\]

We now define
\[
f_n = (1/\lambda_n)A e_n.
\]

Then
\[
\sum_{n=1}^{\infty} \lambda_n (x, e_n) f_n \quad \text{and other proposition will be proved if we Show}
\]
that $\{f_n\}$ is an orthonormal systems.

But
\[
(f_k, f_k) = (\lambda_k^{-1} A e_k, \lambda_k^{-1} A e_k) = \lambda_k^{-1} \lambda_k^{-1} (A^*A e_k, e_k) = (e_k, e_k) = \delta_{kk},
\]

3) Conversely every mapping $A \in \mathcal{L}(H_1, H_2)$ which has a representation (2) with
\[
\lambda_n > 0, \lambda_n \to 0 \text{ is compact.}
\]

Let $A_k$ be $\sum_{n=1}^{\infty} \lambda_n (x, e_n) f_n$, $\|A - A_n\| x \|^2$

$\leq \sum_{n=1}^{\infty} \lambda_n^2 |(x, e_n)|^2$

$n=k+1$

$\leq \epsilon^2 \|x\|^2$, it is $|\lambda_n| \leq \epsilon$ for $n > k(\epsilon)$. 

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Thus $A$ is compact as the limit of $A_n$ in $\ell_b (H_1, H_2)$.

From this proof and (1) follow immediately.

4) Let $H_1$, $H_2$ be Hilbert Spaces. Then every compact $A \in \ell_b (H_1, H_2)$ is the limit of a sequence of mappings of finite rank.

Then $\lambda_n$ of (2) are called the singular values of $A$ and the non-increasing sequence of all singular values of $A$ is uniquely determined by $A$, the representation (2) can be written in a different way using linear forms instead of scalar product for the coefficients of the $f_n$.

The scalar product $(x,y)$ in Hilbert space $H$ is linear in $x$ for $y$ fixed, thus it defines a linear functional, $< \hat{Y}, x > = (x,y)$, where $\hat{Y}$ is uniquely determined. One calls $\hat{Y}$ the Conjugate element to $y$. There exists an Orthonormal basis $\{e_\alpha\}, \alpha \in A$, of $H$ such that

$$x = \sum \xi_\alpha e_\alpha, \quad y = \sum \eta_\alpha e_\alpha$$

$$< \hat{Y}, x > = < \hat{Y}, \sum \xi_\alpha e_\alpha > = \sum \xi_\alpha < \hat{Y}, e_\alpha > = \sum \xi_\alpha \eta_\alpha$$

Since this is true for all $x \in H$, if follows that $\hat{Y} = \sum \eta_\alpha e_\alpha$.

The coefficients of $\hat{Y}$ are the Conjugate of the coefficients of $y$.

Hence the Result.

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