A New Characterization of Bounded Linear Operators on A Hilbert Space

U. K. Srivastava¹, A. U. Talukdar², Md. Shahabuddin³, A. S. Pandit⁴

¹Department of Mathematics, R.S.S. College, Chochahan, P.O.- Aniruddh Belsar, Dist.- Muzaffarpur- 844111, B.R.A. Bihar University, Muzaffarpur-842001, Bihar, India.
²Department of Mathematics, M.G. College, Chalantapara, Bongaigaon, Assam-783382, India.
³Department of Mathematics, P.R.R.D. College, Bairgania, Sitamarhi -843313, B.R.A. Bihar University, Muzaffarpur, -842001, Bihar, India.
⁴Department of Mathematics, B.P.S. College, Desari (Vaishali) – 844504, B.R.A. Bihar University, Muzaffarpur-842001, Bihar, India.

Abstract—This paper presents the study of some important Classes of Bounded linear operators on a Hilbert space including projections, Unitary operators and self-adjoint operators. Here, it is proved in this paper that the theorem of the Riesz representation which characterizes the bounded Linear functionals on a Hilbert Space together with weak convergence in Hilbert spaces.

Keywords—Hilbert Space, Riesz Representation, orthogonality of Projection, The adjoint of an operator, Self-adjoint operator, Unitary operator.

I. INTRODUCTION

E.G. EFFROS (1) and Hall (2) are the pioneer worker of the Present area. In fact, the present work is the extension of work done by Wong, yau - Chen (10), Srivastava et al. (5), Srivastava et al.(6), Srivastava et al. (7), Srivastava et al.(8) and Kumar et al.(9). In this paper we have studied a new Characterization of Bounded Linear Operators on a Hilbert Space.

Here, we use the following definitions, Notations and Fundamental Ideas:

Definition 1: Any projection associated with a direct sum decomposition of a projection on a linear space X is a linear map P:X → X such that P² = P

Definition 2: An orthogonal projection on a Hilbert space H is also a Linear mapping P:H → H satisfying P² = P, <Px, y> = <x, Py> for all x, y ∈ H.

An orthogonal projection is necessarily bounded.

Theorem 1: Let X be a linear space,

(i) If P:X → X is a projection then X = ran P ⊕ kerP

(ii) If X = M ⊕ N where M and N are Linear subspaces of X then there is a projection P:X → X with ran P = M and ker P = N.

Proof:

For (i) We show that x ∈ ran P if x = Px

If x = Px then clearly x ∈ ran P

If x ∈ ran P then x = Py for some y ∈ x

And since P² = P which follows that Px = P²y = Py = x

If x ∈ ran P ∩ kerP then x = Px & Px = 0

So ran P ∩ kerP = {0}. If x ∈ X then

We have x = Px + (x - Px) ; where Px ∈ ran P and (x - Px) ∈ kerP.

Since P (x- Px) = Px - P²x = Px – Px = 0

Thus X = ran P ⊕ kerP. ………………………..(1.1)

Now for (ii)

We consider if X = M ⊕ N then x ∈ X has unique decomposition x = y+z with y ∈ M & z ∈ N and Px = y defines the required Projection.

In particular, in orthogonal subspaces while using Hilbert Space, let us suppose that M is a closed subspace of Hilbert Space H then by well known property we have H = M ⊕ M⊥. We call the projection of H on to M along M⊥ the orthogonal projection of H on to M.

If x = y+z and x₁ = y₁ + z₁ where y, y₁ ∈ M and z, z₁ ∈ M⊥ then by orthogonality of M and M⊥ ⇒ <Px, x₁> = <y, y₁> + <z, y₁> = <y+y+z₁, y₁>

= <x, Px₁> …………… (1.2)

Which states that an orthogonal projection is self Adjoint. We show the properties (1.1) and (1.2) characterize orthogonal projections with Defn-2.

Lemma :- If P is a non zero orthogonal projection then ||P|| = 1.
The norm of bounded linear functional \( \varphi \) is

\[
\| \varphi \| = \sup \| \varphi (x) \|
\]

\[
\| x \| = 1
\]

If \( y \in H \) then \( \varphi_y (x) = < y, x > \) is a bounded Linear functional on \( H \), with

\[
\| \varphi_y \| = \| y \| .
\]

(b) If \( \varphi \) is a bounded Linear functional on a Hilbert space \( H \), then there is a unique vector \( y \in H \) such that

\[
\varphi (x) = < y, x > \quad \text{for all } x \in H.
\]

**Proof.** If \( \varphi = 0 \), then \( y = 0 \), so we suppose that \( \varphi \neq 0 \). In that case, \( \ker \varphi \) is a proper closed subspace of \( H \). and, it implies that, there is a nonzero vector

\[
z \in H \text{ such that } z \perp \ker \varphi. \text{ We define a linear map } P : H \rightarrow H \text{ by}
\]

\[
P x = \varphi (x)/\varphi (z).z
\]

Then \( P^2 = P \), so Theorem 1 implies that, \( H = \ker P \). Moreover,

\[
\ker P = \ker \varphi .
\]

So that \( \ker P \perp \ker \varphi \). It follows that \( P \) is an orthogonal projection, and

\[
H \{ a z | a \in \mathbb{C} \} \oplus \ker \varphi \text{ is an orthogonal direct sum. We can therefore write}
\]

\[
x \in H \text{ as } x = a z + n, \quad a \in \mathbb{C} \text{ and } n \in \ker \varphi .
\]

Taking the inner product of this decomposition with \( z \), we get

\[
a = < z, x >/ \| z \| \| z \|^2, \text{ and evaluating } \varphi \text{ on } x = a z + n, \text{ we find that}
\]

\[
\varphi (x) = a \varphi (z).
\]

The elimination of \( a \) from these equations, and a rearrangement of the result,

yields \( \varphi (x) = < y, x > \), where \( y = \varphi (z)/ \| z \| \|^2. z \).

Thus, every bounded linear functional is given by the inner product with a fixed vector.
We have already seen that $\varphi(x) = \langle y, x \rangle$ defines a bounded linear functional on $H$ for every $y \in H$. To prove that there is a unique $y$ in $H$ associated with a given linear functional, suppose that $\varphi_1 = \varphi_2$. Then $\varphi_1(y) = \varphi_2(y)$. When $y = y_1 - y_2$, which implies that 
$1 \| y_1 - y_2 \|_2^2 = 0$, so $y_1 = y_2$.

The map $J : H \to H^*$ given by $J_x = \varphi_x$, therefore identifies a Hilbert space $H$ with its dual space $H^*$. The norm of $\varphi_x$ is equal to the norm of $y$, so $J$ is an isometry. In this case of complex Hilbert spaces, $J$ is antilinear, rather than linear, because $H$ and $H^*$ are isomorphic as Banach spaces, and anti-isomorphic as Hilbert spaces. Thus Hilbert spaces are special in this respect. This completes the proof of Theorem 2.

**Proposition:** (c) An important consequences of the Riesz representation theorem is the existence of the adjoint of a bounded linear operator on a Hilbert space. The defining property of the adjoint $A^* \in B(H)$ of an operator $A \in B(H)$ is that

\[
\langle x, Ay \rangle = \langle A^*x, y \rangle \quad \text{for all } x, y \in H \quad \ldots \quad (2.2)
\]

The Uniqueness of $A^*$ is obvious. The definition implies that

\[
(A^*)^* = A, \quad (AB)^* = B^*A^*.
\]

To prove that $A^*$ exists, we have to show that for every $x \in H$, there is a vector $z \in H$, depending linearly on $x$ such that

\[
\langle z, y \rangle = \langle x, Ay \rangle \quad \text{for all } y \in H \quad \ldots \quad (2.3)
\]

For fixed $x$, the map $\varphi_x$ defined by $\varphi_x(y) = \langle y, x \rangle A^*$ is a bounded linear functional on $H$, with $\|\varphi_x\| \leq \|A\| \|x\|_2$. By the Riesz representation theorem, there is a unique $z \in H$ such that $\varphi_x(y) = \langle z, y \rangle$. This $z$ satisfies (2.3), so we get $A^*x = z$. The linearity of $A^*$ follows from the uniqueness in the Riesz representation theorem and the linearity of the inner product.

**Definition 3:** A bounded linear operator $A : H \to H$ on a Hilbert space $H$ is self-adjoint if $A^* = A$. Equivalently, a bounded linear operator $A$ on $H$ is self-adjoint if and only if

\[
\langle x, Ay \rangle = \langle Ax, y \rangle \quad \text{for all } x, y \in H.
\]

**Definition 4:** A linear map $U : H_1 \to H_2$ between real or complex Hilbert spaces $H_1$ and $H_2$ is said to be orthogonal or unitary, respectively, if it is invertible and if

\[
\langle Ux, Uy \rangle = \langle x, y \rangle, \quad \text{for all } x, y \in H_1.
\]

Two Hilbert spaces $H_1$ and $H_2$ are isomorphic as Hilbert spaces if there is a unitary linear map between them. Thus a unitary operator is one-to-one and onto, and preserves the inner product. A map $U : H \to H$ is unitary if and only if $U^*U = UU^* = I$.

**Definition 5:** A sequence $(x_n)$ in a Hilbert space $H$ converges weakly to $x \in H$, if $\lim_{n \to \infty} \langle x_n, y \rangle = \langle x, y \rangle$ for all $y \in H$.

Weak convergence is usually written as $x_n \rightharpoonup x$ as $n \to \infty$, to distinguish it from strong, or norm, convergence. From the Riesz representation theorem, this definition of weak convergence for sequences in a Hilbert space is a special case of Definition of weak convergence in a Banach space. Strong convergence implies weak convergence, but the converse is not true on infinite-dimensional spaces.

**Theorem 3:** If $A : H \to H$ is a bounded linear operator, then

\[
\text{ran } A = (\ker A^*)^\perp, \quad \ker A = (\text{ran } A^*)^\perp. \quad \ldots \quad (3.1)
\]

**Proof.** If $x \in \text{ran } A$, there is a $y \in H$ such that $x = Ay$. For any $z \in \ker A^*$, we then have

\[
\langle x, z \rangle = \langle Ay, z \rangle = \langle y, A^*z \rangle = 0
\]

This proves that $\text{ran } A \subset (\ker A^*)^\perp$. Since $(\ker A^*)^\perp$ is closed, it follows that

\[
\text{ran } A \subset (\ker A^*)^\perp. \quad \text{On the other hand, if } x \in (\text{ran } A^*)^\perp, \quad \text{then for all } y \in H \text{ we have}
\]

\[
0 = \langle Ay, x \rangle = \langle y, A^*x \rangle.
\]

Therefore, $A^*x = 0$. This means that $(\text{ran } A^*)^\perp \subset \ker A^*$. By taking the orthogonal complement of this relation, we get

\[
(\ker A^*)^\perp \subset (\text{ran } A)^\perp = \text{ran } A.
\]

Which proves the first part of Theorem 2. To prove the second part, we apply the first part to $A^*$, instead of $A$, use $A^{**} = A$, and take orthogonal complements.
An equivalent formulation of this theorem is that if $A$ is a bounded linear operator on $H$, then $H$ is the orthogonal direct sum

$$H = \text{ran} \ A \bigoplus \ker A^*$$

This completes the proof of the theorem 3.

Thus, from above definitions, theorems, Leema, examples, propositions (a), (b) & (c), which shows Bounded linear functional and Riesz representation on a Hilbert space have the main result as follows :-

**Main result**: “If $(x_n)$ is a sequence in Hilbert space $H$ and $D$ is a dense subset of $H$. Then $(x_n)$ converges weakly to $x$ if and only if:

(a) $\|x_n\| \leq M$ for some constant $M$;

(b) $\langle x_n, y \rangle \to \langle x, y \rangle$ as $n \to \infty$ for all $y \in D^*$.

**Proof of the Main Result**: Suppose that $(x_n)$ is a weakly convergent sequence. We define the bounded linear functional $\varphi_n(x) = \langle x_n, x \rangle$. Then $\|\varphi_n\| = \|x_n\|$. Since $(\varphi_n(x))$ converges for each $x \in H$, it is a bounded sequence, and the uniform boundedness theorem implies that $\{\|\varphi_n\|\}$ is bounded. It follows that a weakly convergent sequence satisfies (a). Part (b) is trivial.

Conversely, suppose that $(x_n)$ satisfies (a) and (b), if $z \in H$, then for any $\varepsilon > 0$ there is a $y \in D$ such that $\|z - y\| < \varepsilon$, and there is an $N$ such that

$1 < |x_n - z| > 1, y \in \varepsilon$ for $n \geq N$. Since $\|x_n\| \leq M$, it follows from the Cauchy-Schwarz inequality that for $n \geq N$

$$|x_n - z| \leq |x_n - x| + |x - z| + |z - y| \leq \varepsilon + \|x_n - x\|\|z - y\| \leq (1 + \|x\|\|z - y\|)\varepsilon.$$

Thus, $x_n - x, z \to 0$ as $n \to \infty$ for every $z \in H$, so $x_n \to x$.

Hence proved.

**Acknowledgement**:

The authors are thankful to prof.(Dr.) S.N. Jha, Ex. Head, Prof. (Dr.) P.K. sharan, Ex. Head, and Prof. (Dr.) B.P Kumar, Present Head of the Deptt. of Mathematics, B.R.A.B.U. Muzaffarpur, Bihar, India and Prof. (Dr.) T.N. Singh, Ex. Head, Ex. Dean(science) and Ex. Chairman, Research Development Council, B.R.A.B.U. Muzaffarpur, Bihar, India, for extending all facilities in the completion of the present research work.

**REFERENCES**


