On The K- Metro Domination Number of Cycle

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Abstract— A dominating set D of a graph \( G = \langle V,E \rangle \) is called metro dominating set of \( G \) if for every pair of vertices \( u, v \) there exists a vertex \( w \) in \( D \) such that \( d(u, w) + d(v, w) \). The k-metro domination number of a cycle \( \gamma_{\beta_k}(C_n) \) is the order of a smallest \( k \)-dominating set of \( C_n \) which resolves as a metric set. In this paper, we calculate the \( k \)-metro domination number of cycles.

Keywords— Domination Number, K-Domination number, Metric Dimension, Land marks, Metro Domination number.

AMS Subject Classification: 05C56.05C38

I. INTRODUCTION

Let \( G(V,E) \) be a graph. A subset of vertices \( D \subseteq V \) is called a dominating set (\( \gamma \)-set) if every vertex in \( V - D \) adjacent to atleast one vertex in \( D \).

The minimum cardinality of a dominating set is called the domination number of the graph \( G \) and is denoted by \( \gamma(G) \).

The Metric Dimension of a graph \( G \) is denoted by \( \beta(G) \), is defined as the cardinality of a minimal subset \( S \subseteq V \) having the property that for each pair of vertices \( u, v \) in \( G \) there exists a vertex \( w \) in \( S \) such that \( d(u, w) + d(v, w) \). The coordinate of each vertex \( v \) of \( V(G) \) with respect of each landmark \( u_i \) belong to \( S \) is defined as usual with \( i^{th} \) component of \( v \) as \( d(u, v_i) \) for each \( i \) and is of dimension \( \beta(G) \).

Metro Domination number introduced by B.Sooryanarayana, Raghunath. [5]. Fink and Ja-cobson [9] , [10] in 1985 introduced the concept of multiple domination. A subset \( D \) of \( V(G) \) is \( K \)-dominating in \( G \) if every vertex of \( V - D \) has at least \( k \) neighbours in \( D \).

The cardinality of minimum \( K \)- dominating set is called \( K \)- domination number \( \gamma_K(G) \).

A dominating set \( D \) of a graph \( G(V,E) \) is called metro dominating set of \( G \) if for each pair of vertices \( u, v \) there exists a vertex \( w \) in \( D \) such that \( d(u, w) + d(v, w) \).

II. OUR RESULTS

Theorem 2.1: For \( n \geq 6, n \neq 10 \)

\[
\gamma_{\beta_2}(C_n) = \left\lceil \frac{n}{5} \right\rceil
\]

Proof. : Let \( v_1, v_2, v_3, \ldots, v_n \) be the vertices of the cycle \( C_n \). Let \( D \) be the minimum 2-dominating set of \( C_n \). Let \( W = V - D \). Now each \( v_i \in W \) is either adjacent to any of the vertex \( D \) or atmost at distance two from atleast one of the vertex of \( D \). Any vertex \( v_k \in D \), will dominates at most 5 vertices including itself. Since the metric dimension of a cycle is two, as in [5], \( D \) also serves as a metric set.

Thus \( \gamma_{\beta_2}(C_n) \geq \left\lceil \frac{n}{5} \right\rceil \) \hspace{1cm} (i)

To prove \( \gamma_{\beta_2}(C_n) \leq \left\lceil \frac{n}{5} \right\rceil \)

We define the 2-dominating set as

Case 1: For \( n \geq 6, n \neq 10 \)

\[ D = \left\{ v_{5k-4}; 1 \leq k \leq \left\lceil \frac{n}{5} \right\rceil \right\} \cup v_n \]

Case 2: \( n = 10 \)

\[ D = \left\{ v_{5k-4}; 1 \leq k \leq \left\lceil \frac{n}{5} \right\rceil \right\} \cup v_n \]

We note that the dominating sets serves as a two dominating set also \( D \) resolve as a metric set.

Thus \( \gamma_{\beta_2}(C_n) \leq \left\lceil \frac{n}{5} \right\rceil \) \hspace{1cm} (ii)

from (i) and (ii)

\[
\gamma_{\beta_2}(C_n) = \left\lceil \frac{n}{5} \right\rceil
\]
Figure 1

Case 1:

\[ \gamma_{\beta_2}(C_5) = 2 \]

\[ D = \{v_1\} \] will be the 2-dominating set but which is not serves as a metric set \( \beta(C_n) = 2 \) hence we have to choose one more vertex as a land mark show in to achieve metric set figure 1a,

Now \[ D_1 = \{v_1, v_5\} \] serves as both 2-dominating and metric set.

Thus \[ \gamma_{\beta_2}(C_5) = 2 \]

Case 2:

\[ \gamma_{\beta_2}(C_{15}) = 3 \]

Theorem 2.2: For \( n \geq 8, n \neq 14 \)

\[ \gamma_{\beta_2}(C_n) = \left \lfloor \frac{n}{7} \right \rfloor \]

Proof: Let \( v_1, v_2, v_3, \ldots, v_{15} \) be the vertices of the cycle \( C_{15} \). Let \( D \) be the minimum 3-dominating set of \( C_{15} \). Let \( W = V - D \), Now each \( v_i \in W \) is either adjacent to any of the vertex \( D \) are atleast the distance three from atleast one of the vertex of \( D \). So any vertex \( v_k \in D \), will dominates atmost 7 vertices. Since metric dimension of a cycle is 2 [5].

\[ D \] also serves as a metric set.

Thus

\[ \gamma_{\beta_2}(C_{15}) \geq \left \lfloor \frac{n}{7} \right \rfloor \]

(i)

To prove \[ \gamma_{\beta_2}(C_n) \leq \left \lfloor \frac{n}{7} \right \rfloor \]
We define the 3-dominating set

**Case 1:** $n \geq 8, n \neq 14$

$$D = \{v_{k-6}; 1 \leq k \leq \lceil \frac{n}{7} \rceil \}$$

**Case 2:** $n = 14$.

$$D = \{v_{k-6}; 1 \leq k \leq \lceil \frac{n}{7} \rceil \cup v_n \}$$

We note that the dominating sets serves as a metric set.

Thus

$$\gamma_{\beta_3}(C_n) \leq \left\lceil \frac{n}{7} \right\rceil$$  \hspace{1cm} (ii)

from (i) and (ii)

$$\gamma_{\beta_3}(C_n) = \left\lceil \frac{n}{7} \right\rceil$$

**C6:**

$$D = \{v_1\}$$ will be the 3-dominating set but which is not serves as a metric set. Hence we have to choose one more vertex as a land mark as shown in figure 4a, to achieve metric set.

Now $D_1 = \{v_2, v_6\}$ serves as both 3-dominating and metric set.

Thus $\gamma_{\beta_3}(C_6) = 2$.

**Case 1:**

**C22:**

$$\gamma_{\beta_3}(C_{22}) = 4$$

**Case 2:** $n = 14$, $C_{14}$:

$$D = \{v_1, v_6\}$$ will be the minimum 3-dominating set but which does not serves as a metric set, because $v_1$ and $v_6$ are antipodal vertices. Hence we have to choose one more vertex as a land mark as shown in figure 6a.

Now $D_1 = \{v_1, v_3, v_{14}\}$ serves as both 3-dominating and metric set.

Thus $\gamma_{\beta_3}(C_{14}) = 3$.

**Theorem 2.3:** For $n \geq 10, n \neq 18$.

$$\gamma_{\beta_4}(C_n) = \left\lceil \frac{n}{9} \right\rceil$$

**Proof:** Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of the cycle $C_n$. Let $D$ be the minimum 4-dominating set of $C_n$. Let $W = V - D$. Now each $v_i \in W$ is either adjacent to any of the vertex $D$ or atmost the distance four from at least one of the vertex of $D$. So any vertex $v_k \in D$ will dominates atmost 9 vertices. Since metric dimension of a cycle is 2. $D$ also serves as a metric set.

Thus

$$\gamma_{\beta_4}(C_n) \geq \left\lceil \frac{n}{9} \right\rceil$$  \hspace{1cm} (i)
To prove
\[
\gamma_{\delta_4}(C_n) \leq \left\lceil \frac{n}{9} \right\rceil
\]
We define the 4-dominating set

Case 1: \( n \geq 10, n \neq 14 \)
\[ D = \left\{ v_{9k+b} : 1 \leq k \leq \left\lceil \frac{n}{9} \right\rceil \} \]

Case 2: \( n = 14 \)
\[ D = \left\{ v_{9k+b} : 1 \leq k \leq \left\lceil \frac{n}{9} \right\rceil \} \cup v_n \]

We note that D serves as a 4-dominating set, also D resolve as a metric set
Thus
\[
\gamma_{\delta_4}(C_n) \leq \left\lceil \frac{n}{9} \right\rceil
\]

from (i) and (ii)
\[
\gamma_{\delta_4}(C_n) = \left\lceil \frac{n}{9} \right\rceil
\]

Cs:

Now \( D_1 = \{ v_1, v_3 \} \) serves as both 4-dominating and metric set.
Thus
\[
\gamma_{\delta_4}(C_9) = 2
\]

C9:

\[
D = \{ v_1 \}, \text{ is 4-dominating set which is not a metric set.}
\]
Therefore \( D = \{ v_1, v_3 \} \) serves as both 4-dominating and metric dimension set.
Thus
\[
\gamma_{\delta_4}(C_9) = 2
\]

Case 1: \( n \geq 10, n \neq 18 \)

C25:

\[
\gamma_{\delta_4}(C_{25}) = 3
\]
Case 2: \( n = 18 \)

\[ C_{18} : \]

\[ D = \{ v_1, v_{10} \} \] will be the minimum 4-dominating set but which does not serves as a metric set, because \( v_1 \) and \( v_{10} \) are antipodal vertices. Hence we have to choose one more vertex as a landmark as shown in figure 10a.

Now \( D_1 = \{ v_1, v_{10}, v_{18} \} \) serves as both 4-dominating and metric dimension set.

Thus \( \gamma_{\beta_5}(C_{18}) = 3 \)

**Theorem 2.4:** For \( n \geq 12, n \neq 22 \)

\[ \gamma_{\beta_5}(C_n) = \left\lceil \frac{n}{11} \right\rceil \]

**Proof:** Let \( v_1, v_2, v_3, \ldots, v_n \) be the vertices of the cycle \( C_n \). Let \( D \) be the minimum 5-dominating set of \( C_n \). Let \( W = V - D \). Now each \( v_i \in W \) is either adjacent to any of the vertex \( D \) or at most at the distance five from at least one of the vertex of \( D \). So any vertex \( v_k \in D \), will dominates at most 10 vertices. Since metric dimension of a cycle is 2. \( D \) also serves as a metric set.

Thus \( \gamma_{\beta_5}(C_n) \geq \left\lceil \frac{n}{11} \right\rceil \) \hspace{1cm} (i)

To prove

\[ \gamma_{\beta_5}(C_n) \leq \left\lceil \frac{n}{11} \right\rceil \]

We define the 5-dominating set

**Case 1:** \( n \geq 12, n \neq 22 \)

\[ D = \{ v_{11k-10} : 1 \leq k \leq \left\lfloor \frac{n}{11} \right\rfloor \} \]

**Case 2:** \( n = 22 \)

\[ D = \{ v_{11k-10} : 1 \leq k \leq \left\lfloor \frac{n}{11} \right\rfloor \} \cup v_n \]

We note that \( D \) serves as a 5-dominating set, also \( D \) resolve as a metric set.

Thus

\[ \gamma_{\beta_5}(C_n) \leq \left\lfloor \frac{n}{11} \right\rfloor \] \hspace{1cm} (ii)

from (i) and (ii)

\[ \gamma_{\beta_5}(C_n) = \left\lfloor \frac{n}{11} \right\rfloor \]

**C_{10}**:

\[ D = \{ v_1 \} \], will be the 5-dominating set but which is not serves as a metric set. Hence we have to choose one more vertex as a landmark shown in figure 11a, to achieve metric set.

Now \( D_1 = \{ v_1, v_{10} \} \) serves as both 5-dominating and metric set.

Thus \( \gamma_{\beta_5}(C_{10}) = 2 \)
Case 1: \( n \geq 12, n \neq 22 \)

\[ y_{\beta_2}(C_{16}) = 2 \]

Case 2: \( n = 22 \)

\[ y_{\beta_2}(C_{22}) = 3 \]

**Theorem 2.4:** For \( n \geq 2(k+1), n \neq 4k+2 \)

\[ y_{\beta_2}(C_n) = \left\lfloor \frac{n}{2k+1} \right\rfloor \]

Proof follows from the generalisation of theorem 1, theorem 2, theorem 3 and theorem 4

**REFERENCES**


