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Necessary and Sufficient Conditions for Solution of the Fourth order Cauchy Difference Equation on Finite Cyclic Groups

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Abstract— Let f: G → H be a function, where (G, ·) is a group and (H, +) is an abelian group. In this paper, the following Fourth Order Cauchy difference of

\[ f: C^n f(x_1, x_2, x_3, x_4, x_5) = f(C_1(\prod_{j=1}^{5} x_j)) \]

is studied. Where \( f(C_r(\prod_{j=1}^{n} x_j)) \) is defined as function of combination \( r \) at a time from \( n \) objects. Then Necessary and Sufficient conditions on finite cyclic groups are obtained.

Keywords— Cauchy difference equation; finite cyclic group

I. INTRODUCTION

It is well known from [1] that Jenson’s functional equation \( f(x + y) + f(x - y) = 2 f(x) \) (1.1) with additional condition \( f(0) = 0 \), is equivalent to Cauchy’s equation \( f(x + y) = f(x) + f(y) \) on the real line. Let \((G, .)\) be a group, \((H, +)\) be an abelian group. Let \( e \in G \) and \( 0 \in H \) denote identity elements. The study of (1.1) was extended groups for \( f \) maps \( G \) into \( H \) in [2], where the general solution for a free group \( H \) with two generators and \( G = GL_n(z), n \geq 3 \) (see[3]). Since the functional equations involve Cauchy difference, which made it become much more interesting [4–7]. For a function \( f: G \rightarrow H \), its cauchy difference \( C^{(m)} f \), is defined by

\[ C^{(0)} f = f, \]

\[ C^{(1)} f(x_1, x_2) = f(x_1, x_2) - f(x_1) - f(x_2) \] (1.3)

\[ C^{(m+1)} f(x_1, x_2, \ldots, x_{m+2}) = C^{(m)} f(x_1, x_2, x_3, \ldots, x_{m+2}) - C^{(m)} f(x_1, x_3, \ldots, x_{m+2}) \] (1.4)

The first order cauchy difference \( C^{(1)} f \) will be abbreviated as \( Cf \). In [9], by using the reduction formulas and relations, as given in [2,3], the general solution of third order Cauchy difference equation was provided on symmetric groups.

In this paper, we consider the following functional equation:

\[ f(C_5(\prod_{i=1}^{5} x_i)) - f(C_4(\prod_{i=1}^{4} x_i)) + f(C_3(\prod_{i=1}^{3} x_i)) \]

\[ - f(C_2(\prod_{i=1}^{2} x_i)) + f(C_1(\prod_{i=1}^{1} x_i)) = 0 \forall x_1, x_2, x_3, x_4, x_5 \in G \] (1.5)

It follows from (1.4) that (1.5) is equivalent to the vanishing fourth order cauchy difference equation \( C^{(4)} f = 0 \) The purpose of this paper is to determine the solutions of equation (1.5). The solution of equation (1.5) will be denoted by

\[ KerC^{(4)}(G, H) = \{ f : G \rightarrow H \mid f satisfies (1.5) \} \] (1.6)

Remark 1. \( KerC^{(4)}(G, H) \) is an abelian group under the pointwise addition of functions;

2. \( Hom(G, H) \subseteq KerC^{(4)}(G, H) \)

II. PROPERTIES OF SOLUTIONS

Lemma 1 Suppose that \( f \in KerC^{(4)}(G, H) \). Then

\[ f(e) = 0, \] (2.1)

\[ Cf(x, y) = 0, \text{ when } x = e \text{ or } y = e \] (2.2)

\[ C^{(2)} f(x, y, z) = 0, \text{ when } x = e \text{ or } y = e \text{ or } z = e \] (2.3)

\[ C^{(3)} f(x, y, z, u) = 0, \text{ when } x = e \text{ or } y = e \text{ or } z = e \text{ or } u = e \] (2.4)

\( C^{(3)} f \) is a homomorphism w.r.t. each variable

\[ f(x^a) = nf(x) + nC_1 \quad Cf(x) + nC_2 \quad C^{(2)} f(x, x, x) \]

\[ + nC_3 \quad C^{(3)} f(x, x, x, x) \] (2.6)

for all \( x, y, z, u \in G \) and \( n \in Z \).
Proof 1: Putting $x_i = e$ in (1.5) we get (2.1).

$$C^3 f(x, y, z, u) = f(xyzw) - f(xyz) - f(xywu) + f(xzu) - f(yzu) + f(xyzw) + f(xywu) + f(xzu) - f(xwz) - f(yzu) - f(xwz) + f(xywu)$$

and

$$C^3 f(x, y, z, u) + C^3 f(x, w, z, u)$$

One can easily check that

$$C^3 f(x, y, z, u) - C^3 f(x, y, z, u) - C^3 f(x, w, z, u) = C^4 f(x, y, w, z, u) = 0$$

Hence, the above relations imply the $C^3 f(x, z, u)$ is a homomorphism. Similarly, the fact is also true for
C^{(3)} f (., y, z, u) , \quad C^{(3)} f (x, y, u, u) \quad \text{and} \quad C^{(3)} f (x, y, z, u) . \ \text{This proves (2.5).}

We now consider (2.6). Actually, it is trivial for $n = 0, 1, 2, 3$ by (2.1) and by the definition of $C_f$. Suppose that (2.6) holds for all natural numbers smaller than $n \geq 5$, then

$$f(x^n) = f(x^{n-3}xxx)$$

$$= f(x^{n-3}xx) + f(x^{n-3}xx) + f(x^{n-3}xx) - f(x^{n-3}x)$$

$$- f(x^{n-3}x) - f(x^{n-3}x) - f(xx) - f(xx)$$

$$+ f(x^{n-3}) + f(x) + f(x) + C^{(3)} f(x^{n-3}, x, x, x)$$

$$= f(x^{n-1}) + f(x^{n-1}) + f(x^{n-1}) + f(x^3) - f(x^{n-2}) - f(x^{n-2})$$

$$- f(x^{n-2}) - f(x^2) - f(x^2) + f(x^3) + f(x) + f(x) + f(x)$$

$$+ C^{(3)} f(x^{n-3}, x, x, x)$$

$$= 3 f(x^{n-1}) + f(x^3) - 3 f(x^{n-2}) - 3 f(x^2) + f(x^{n-3}) + 3 f(x)$$

$$= [3(n-1) f(x) + (n-1) C_2 C_f(x, x) + (n-1) C_2 C^{(2)} f(x, x, x)$$

$$+ (n-1) C_3 C^{(3)} f(x, x, x, x)]$$

$$+ [3 f(x) + 3 C_2 C_f(x, x) + 3 C_3 C^2 f(x, x, x)]$$

$$- 3[(n-2) f(x) + (n-2) C_2 C_f(x, x) + (n-2) C_2 C^{(2)} f(x, x, x)$$

$$+ (n-2) C_4 C^{(3)} f(x, x, x, x)] - 3[2 f(x) + 2 C_2 C_f(x, x)]$$

$$+ [(n-3) f(x) + (n-3) C_2 C_f(x, x) + (n-3) C_2 C^{(2)} f(x, x, x)$$

$$+ (n-3) C_3 C^{(3)} f(x, x, x, x)] + [3 f(x) + (n-3) C^{(3)} f(x, x, x, x)]$$

$$= nf(x) + n C_2 C_f(x, x) + n C_3 C^{(2)} f(x, x, x)$$

$$+ n C_4 C^{(3)} f(x, x, x, x)$$

where the definition of $C^{(3)} f$ and (2.5) are used in the second equation. This gives (2.6) for all $n \geq 0$. On the other hand, for any fixed integer $n > 0$, by (1.4) and (2.1), we have

$$C^{(3)} f(x^n, x^n, x^n) = f(x^{2n}) - f(x^n) - f(x^{2n}) - f(x^n)$$

$$+ f(e) + f(x^{2n}) + f(x^{2n}) + f(e) + f(e)$$

$$+ f(x^{2n}) - f(x^n) - f(x^{2n}) - f(x^n)$$

$$= 4 f(x^{2n}) - 6 f(x^n) - f(x^{2n})$$

$$\Rightarrow f(x^n) = 4 f(x^{2n}) - 6 f(x^n) - f(x^{3n})$$

$$- C^{(3)} f(x^n, x^{2n}, x^n, x^n)$$

$$= 4[2nf(x) + 2n C_2 C_f(x, x) + 2n C_3$$

$$C^{(2)} f(x, x, x) + 2n C_4 C^{(3)} f(x, x, x, x)]$$

$$- 3nf(x) + 3n C_2 C_f(x, x) + 3n C_3$$

$$C^{(2)} f(x, x, x) + 3n C_4 C^{(3)} f(x, x, x, x)]$$

$$+ n^4 C^{(3)} f(x, x, x, x)$$

$$= -nf(x) + n C_2 C_f(x, x) + n C_3$$

$$C^{(2)} f(x, x, x) + n C_4 C^{(3)} f(x, x, x, x)]$$

From (2.5) and the above claim for $n > 0$. This confirms (2.6) for $n < 0$.

Remark 2 For any function $f: G \to H$, the following statements are pairwise equivalent:

The function $f \in Ker C^{(4)}(G, H)$;

- $C^{(3)} f (., y, z, u)$ is a homomorphism;
- $C^{(3)} f (x, ., u)$ is a homomorphism;
- $C^{(3)} f (x, y, ., u)$ is a homomorphism;
- $C^{(3)} f (x, y, z, .)$ is a homomorphism;
Before presenting Proposition 1, we first introduce the following useful lemma, which was given in [8]

Lemma 2 (Lemma 2.4 in [8]) The following identity is valid for any function \( f : G \to H \) and \( l \in \mathbb{N} \):

\[
f(x_1, x_2, \ldots, x_l) = \sum_{m \leq l} \sum_{i \in \{1, \ldots, m\}} C^{(m-1)} f(x_1, x_2, \ldots, x_m) \quad (2.7)
\]

**Proposition 1** Suppose that \( f \in \text{Ker}C^4(G, H) \). Then

\[
f(x_1, x_2, \ldots, x_l) = \sum_{i \in \{1, \ldots, l\}} n_i f(x_i) + n_2 C^2 f(x_1, x_2) + n_3 C^3 f(x_1, x_2, x_3) + \sum_{l \leq j} C^4 f(x_j, x_1, x_2, x_3)
\]

The vanishing of \( C^{(m-1)} f \) for \( m \geq 5 \) yields

\[
f(x_1, x_2, \ldots, x_l) = \sum_{i \in \{1, \ldots, l\}} n_i f(x_i) + \sum_{l \leq j} C^2 f(x_1, x_2) + n_3 C^3 f(x_1, x_2, x_3)
\]

\[
+ \sum_{l \leq j} C^4 f(x_j, x_1, x_2, x_3) + \sum_{l \leq j} C^5 f(x_j, x_1, x_2, x_3, x_4)
\]

Therefore, by (2.6) and (2.5), we have

\[
f(x_1, x_2, \ldots, x_l) = n_1 f(x_1) + n_2 C_2 f(x_1, x_2) + n_3 C_3 f(x_1, x_2, x_3) + n_4 C_4 f(x_1, x_2, x_3, x_4)
\]

**III. Solution on the Finite Cyclic Group** \( C_n \)

Let \( C_n = \langle a \mid a^n = e \rangle \) be a cyclic group of order \( n \) with generator \( a \). In this section, we study the general solution on the finite cyclic group \( C_n \).

**Theorem 1** Assume that \( n \) is odd and \( nCf(a, a) = 0 \) and \( nC^2 f(a, a, a) = 0 \). Then \( f \in \text{Ker}C^4(C_n, H) \) if and only if it is given by

\[
f(a^p) = pf(a) + pC_2 f(a, a) + pC_3 f(a, a, a) + pC_4 C^5 f(a, a, a, a) \quad \forall p \in \mathbb{Z} \quad (3.1)
\]

where \( f(a) \) and \( C^3 f(a, a, a) \) satisfy

\[
nCf(a, a, a, a) = 0 \quad (3.2)
\]

\[
nf(a) + nC_4 C^3 f(a, a, a, a) = 0 \quad (3.3)
\]

**Proof 3** Necessity. Let \( f : C_n \to H \) be a function satisfying (1.5). Then by (2.6), we see that \( f \) also satisfies (3.1) i.e.,

\[
f(a^p) = pf(a) + pC_2 f(a, a) + pC_3 f(a, a, a) + pC_4 C^3 f(a, a, a, a) \quad \forall p \in \mathbb{Z}
\]
Let $p = n$ in (3.1), since $nCf(a, a) = 0$ and $nC^2f(a, a, a) = 0$ and $nC_2$ and $nC_3$ are integers, the summand $nC_2$ $C(a, a)$ $= 0$ and $nC_3$ $C^2(a, a, a) = 0$ and by the fact that $a^n = e$, $f(e) = 0$, we obtain

$$nf(a) + nC_4 C^3(a, a, a, a) = 0$$

Furthermore, by using (2.5), (2.4), and $a^n = e$, we get

$$nC^3f(a, a, a, a) = C^3f(e, e, e, e) = 0$$

This proves (3.2)-(3.3). Sufficiency. We claim that (3.1)-(3.3) defines a function on $C_n$. Indeed, for each $p \in Z$, by (3.1) and (3.3) we have

$$f(a^{n+p}) - f(a^n) = [(p+n)f(a) + (p+n)C_2 Cf(a, a) + (p+n)C_3 C^2f(a, a, a) + (p+n)C_4 C^3f(a, a, a, a)] - pC_2 Cf(a, a) + pC_3 C^2f(a, a, a) + pC_4 C^3f(a, a, a, a) = (p+n)C_2 Cf(a, a) + (p+n)C_3 C^2f(a, a, a) + (p+n)C_4 C^3f(a, a, a, a) = 0$$

Finally, for any $x = a^m, y = a^n, z = a^q, w = a^r, u = a^s \in C_n$ we have

$$f(xy) + f(xz) + f(zy) - f(zy) - f(xy) + f(zy) + f(xy) = f(xz) + f(zy) + f(xy)$$

$$f(xw) + f(yz) + f(wz) = f(xz) + f(zy) + f(xy)$$

Here the last identity is obtained because $a^n = e$, $f(e) = 0$, and $n$ is odd.
which, after a long and tedious computation, gives \( f \). Consequently, \( f \in C^4(S^n, H) \). This completes the proof.

REFERENCES


