Almost Generalized Principally Injective Modules

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Abstract— The right R module M is defined such a way that, if for \( 0 \neq a \in R \), there is a positive integer \( n = n(a) \) and submodule \( X_a \) of \( M \) such that
\[
I_M r_R(a^n) = Ma^n + X_a
\]
as left module, where \( a^n \neq 0 \) then \( M \) is called Almost Generalized. Principally injective module. In this paper we define and prove the parallel result of Zhao Yue, we also prove that, in a ring \( R \) whose every simple singular right \( R \) module is AGP injective, then \( J(R) \) contains no non zero nil potent elements
If and only if \( J(R)=0 \).

Keywords-- AGP injective, von Neumann regular Ring, nilpotent element, Reduce, ZI Ring, quasi duo, strong regular ring.

I. INTRODUCTION

R is an associative ring with unity and all modules are unitary \( R \)-modules. Here \( J(R) \), \( Z_r(R) \) for Jacobson radical and right singular ideal resp. let \( M \) be a right \( R \)-Module with \( S = \text{End}_R(M) \), \( X \leq M, A \leq S, B \leq R \) denote the submodules.

Then
\[
I_s(X) = \{ s \in S : sx = 0, \forall x \in X \},
\]
\[
r_M(A) = \{ m \in M : am = 0, \forall a \in A \}
\]
\[
l_M(B) = \{ m \in M : mb = 0, \forall b \in B \}.
\]

If \( \{a\} \subseteq R \), \( r_M(a) \), \( l_M(a) \) denote the right and left annihilator of \( a \), respectively.

In mathematics, more specifically ring theory, a branch of abstract algebra, the Jacobson radical of a ring \( R \) is the ideal consisting of those elements in \( R \) that annihilate all simple right \( R \)-modules. It happens that substituting “left” in place of “right” in the definition yields the same ideal, and so the notion is left-right symmetric. The Jacobson radical of a ring is frequently denoted by \( J(R) \) or \( \text{rad}(R) \); however to avoid confusion with other radicals of rings, the former notation will be preferred in this article. The Jacobson radical is named after Nathan Jacobson, who was the first to study it for arbitrary rings in (Jacobson 1945).

The Jacobson radical of a ring has numerous internal characterizations, including a few definition that successfully extend the notion to rings without unity. The radical of a module extends the definition of the Jacobson radical to include modules. The Jacobson radical plays a prominent role in many ring and module theoretic results, such as Nakayama’s lemma.

A right \( R \)-module \( M \) is called GP-injective if for any \( 0 \neq a \in R \), there exists \( n \geq 1 \) such that \( a^n \neq 0 \) and
\[
l_M r_R(a^n) = M_{a^n} \cdots
\]
A right \( R \)-module \( M \) is AGP-injective if for any \( a \in J(R) \), there exists an \( S \)-sub module \( X_a \) of \( M \) such that \( l_M r_R(a^n) = M_{a^n} \oplus X_{a^n} \). A ring \( R \) is called right AGP-injective module if \( R \) is an AGP-injective module. AGP-injective ring need not be right GP-injective. APS-injective rings are the proper generalization of \( PS \)-injective rings and \( AP \)-injective rings. There are many similarities between \( AP \)-injective rings and \( APS \)-injective rings.

Definition:

Let \( M \) be a right \( R \)-module, \( S = \text{End}_R(M) \). The module \( M \) is called almost generalized principally small injective module if \( (\text{AGP}-\text{S-injective}) \) [11] if for any \( a \in J(R) \), there exists \( n \geq 0 \) and left sub module \( X_a \) of \( M \) such that \( l_M r_R(a^n) = M_{a^n} \oplus X_{a^n} \) as left \( S \)-modules. If \( R \) is an \( A \)-injective module, then we call \( R \) is a right \( A \)-injective ring.

Lemma:1.1 Let \( M \) be a right \( R \)-module with \( S = \text{End}_R(M) \). If
\[
l_M r_R(a^n) = M_{a^n} \oplus X_{a^n},
\]
where \( X_a \) is a left \( S \)-submodule of \( M_R \). The map
\[
\phi : a^n R \to M \quad \text{a right } R \text{-homomorphism},
\]
then
\[
f(a^n) = ma^n + x \quad \forall m \in M, x \in X_a
\]

Proof:

Since
\[
\phi(a^n) r_R(a^n) = \phi[a^n r_R(a^n)] = \phi(0) = 0.
\]
Therefore
\[
r_R(a^n) \subseteq \phi r_R(f(a^n)).
\]
Thus
\[
l_M r_R(\phi(a^n)) \subseteq l_M r_R(a^n) = M_{a^n} \oplus X_{a^n}
\]
and
\[
\phi(a^n) \in l_M r_R(a^n) \Rightarrow f(a^n)
\]
\[
= M_{a^n} \oplus X_{a^n} = ma^n + x \quad \forall m \in M_R, x \in X_{a^n}.
\]
Lemma 1.2. Let M be a right R-module with $S = \text{End}_R(M)$. If $f(a^n) = ma^n + x \forall m \in M, x \in X_{a^n}$ as left S-submodule. Then $I_{R/M}r_R(\phi(a^n)) = M_{a^n} \oplus X_{a^n}$ where $X_{a^n} = \{ \phi \in \text{Hom}_R(a^n R, M): \phi(a^n) \in X_{a^n} \}$. \\

Definition: A ring R is called quasi-duo [10] if every maximal right ideal of R is two sided ideal. A ring R is called MELT, if every essential right (left) ideal of R is two sided ideal. A ring R is called right weakly continuous if $J(R) = Z_r(R), R/J(R)$ is regular and idempotent can be lifted modulo J(R). A ring R is called weakly regular if $I^2 = I$ for each right (left) ideal I of R, equivalently for every $a \in R$, $a \in RaRa$ (a $\in aRaR$). R is weakly regular if it is both left and right weakly regular.

Theorem 1.1. Let R be a quasi-duo, then the following statements are equivalent:
1) R is von-Neumann regular.
2) R is right weakly continuous ring whose simple singular right R-modules are GAP-S- injective.

Proof: (1) $\Rightarrow$ (2)
Since R is von-Neumann regular, then every right R module is AGP-S-injective. (2) $\Rightarrow$ (1) suppose that $Z_r(R)$ is not reduced ($Z_r(R) \neq 0$). Then there exists non-zero $a \in Z_r(R)$ such that $a^2 = 0$. Therefore $Z_r(R) + r(a) = R$. Let $Z_r(R) + r(a) \neq R$ there exists a maximal essential right ideal M containing $Z_r(R) + r(a)$. Thus $\frac{R}{M}$ is AGP-S-injective, then $I_{R/M}r_R(a^n) = (R/M)a^n \oplus X_{a^n}, X_{a^n} \leq R/M$. The map $f: a^n R \rightarrow R/M$ defined by $f(a^n r) = r + M$. By above lemma $1 + M = f(a^n r) = ta^n + M + x, t \in R, x \in X_{a^n}$, $1 - ta^n + M = x \in R/M \cap X_{a^n} = 0$, since R is a semi prime ring, so $1 - ta^n \in M, ta^n \in M$.

Hence $1 \in M$, which is contradiction. Thus $Z_r(R) + r(a) = R$. Thus we can write $1 = c + d$ for some $c \in Z_r(R)$ and $d \in r(a^n)$. Thus $a^n = ca^n$ and so $(1 - c)a^n = 0$. Since $c \in Z_r(R) = J(R), 1 - c$ is invertible. Thus $a^n = 0$, which is contradiction. Therefore $Z_r(R)$ is reduced and so $Z_r(R) = 0$. \[
\]

Theorem 1.2. If every simple right R-module is GAP-S-injective, then R is a right weakly regular ring.

Proof: For every $0 \neq a^n \in R$ we claim that $Ra^n R + r(a^n) = R$. Suppose that $b^n \in R$ such that $Rb^n R + r(b^n) \neq R$, and let M be a maximal right ideal containing $Rb^n R + r(a^n)$. Thus $R/M$ is AGP-S-injective, the $I_{R/M}r_R(b^n) = (R/M)b^n \oplus X_{b^n}, X_{b^n} \leq R/M$. Let $f: b^n R \rightarrow R/M$ defined by $f(b^n r) = r + M$. If $b^n r_1 = b^n r_2 \Rightarrow b^n (r_1 - r_2) = 0 \Rightarrow (r_1 - r_2) \in r(b^n) \subset M$. Hence $r_1 + M = r_2 + M$. So f is well defined. Then $1 + M = f(b^n) = b^n + M + x, t \in R, x \in X_{a^n}$. $1 - tb^n + M = x \in R/M \cap X_{b^n} = 0$, $1 - tb^n \in M, tb^n \in M$ and hence $1 \in M$, which is contradiction. Therefore $Ra^n R + r(a^n) = R$. \[
\]

Definition: R is a ZI ring if for $a, b \in R, ab = 0 \Rightarrow aRb = 0$. A ring R is called idempotent reflexive if $a Re = 0 \Rightarrow cRa = 0$ for $a, e = e^2 \in R$. A Ring R is called local ring if it has one maximal ideal. A ring R is local ring if and only if the set of all non-invertible element of R is an ideal of R. If S is local, then $J(S) = \{ s \in S: ker s \neq 0 \}$.

Corollary 1.1. Let R be a ZI ring. If every simple right R-module is AGP-S-injective, then R is weak regular.

Proof: [11]

Theorem 1.3. Let R be an idempotent reflexive ring. If every simple singular right R-module is GAP-S-injective, then R is right weakly regular.
Proof:[11]

**Definition:**

A ring $R$ is called strongly regular if for every $a \in R$, there exists $b \in B$ such that $a = a^2 b; R$ is called reduced if it has no nonzero nilpotent elements. Clearly a ring $R$ is reduced if and only if $r(a^k) = r(a)$ for any $a \in R$. A reduced ring $R$ is AGP-injective if and only if it is AGP-S-injective.

**Theorem:1.4.** If $R$ is right quasi-duo [ME], then the following statements are equivalents:

1. Every right $R$-module is AGP-S-injective.
2. Every cyclic right $R$-module is AGP-S-injective.
3. Every simple right $R$-module is AGP-S-injective.
4. $R$ is strongly regular ring.
5. $R$ is von-Neumann regular ring.
6. $R$ is a right weakly regular ring.
7. $R$ is local ring.

**Proof:** These are clear (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (5) $\Rightarrow$ (1)

(4)$\Rightarrow$(6) For every $0 \neq a^n \in R$ we claim that $a^n R = R$. Let $a^n R + r(a^n) \neq R$. $M$ be a maximal right ideal containing $a^n R + r(a^n)$. Since $R/M$ is simple AGP-S-injective. Thus $R/M$ is AGP-injective, the $l_{R/M} r_k(a^n) = (R/M) a^n \oplus X_a$, $X_a^* \leq R/M$. Let $f : a^n R \to R/M$ defined by $f(a^n r) = r + M$. If $a^n r_i = a^n r_j \Rightarrow a^n (r_i - r_j) = 0 \Rightarrow (r_i - r_j) \in R$ and hence $r_i + M = r_j + M$. So $f$ is well defined. Then $1 + m = f(a^n) = ta^n + M + x$, $t \in R, x \in X_a^*$. $1 - ta^n + M = x \in \frac{R}{M} \cap X_a^* = 0$,

(6)$\Rightarrow$(7) last part theorem 1.1. //

**Lemma:1.3.** If $R$ is a ring whose every simple singular right $R$-module is AGP-S-injective, then $J(R) \cap Z(R)$ contains no nonzero nilpotent elements.

**Proof:** Consider $b \in J(R) \cap Z(R)$ with $b^2 = 0$, then $r(b^n) + Rb^n R$ is an essential right ideal of $R$. We claim that $Ra^n R + r(a^n) = R$. Suppose that $b^n \in R$ such that $Rb^n R + r(b^n) \neq R$, and let $M$ be a maximal essential right ideal containing $Rb^n R + r(a^n)$. By Assumption $R/M$ is AGP-S-injective, thus $l_{R/M} r_k(b^n) = (R/M) b^n \oplus X_b$, $X_b^* \leq R/M$. Let $f : b^n R \to R/M$ defined by $f(b^n r) = r + M$. If $b^n r_1 = b^n r_2 \Rightarrow b^n (r_1 - r_2) = 0 \Rightarrow (r_1 - r_2) \in R(b^n)$ and hence $r_1 + M = r_2 + M$. So $f$ is well defined. Then $1 + M = f(b^n) = tb^n + M + x$, $t \in R, x \in X_b^*$. $1 - tb^n + M = x \in \frac{R}{M} \cap X_b^* = 0$,

(6)$\Rightarrow$(7) last part theorem 1.1. //
Theorem 1.6. If $R$ is a ring whose every simple singular right $R$-module is AGP-S-injective, then $J(R)$ contains no nonzero nilpotent elements if and only if $J(R) = 0$.

Proof: Assume that $J(R)$ contains nonzero nilpotent elements. For any $b \in (J(R))$, if $L = R$, then $J(R) = 0$. If $L \neq R$, then there exists a right ideal $K$ of $R$ such that $L \oplus K$ is an essential right ideal of $R$. We claim that $L \oplus K = R$. Suppose that $L \oplus K \neq R$. There exists a maximal right ideal $M$ of $R$ containing $L \oplus K$. By assumption the simple singular right $R$-module $R/M$ is AGP-S-injective, thus $I_{R/M}a^n = (R/M)b^n \oplus X$, $X \subseteq R/M$.

Let $f : b^nR \rightarrow R/M$ defined by $f(b^n) = r + M$. If $b^n r_1 = b^n r_2 \Rightarrow b^n(r_1 - r_2) = 0 \Rightarrow (r_1 - r_2) \in r(b^n) \subseteq M$.

Hence $r_1 + M = r_2 + M$. So $f$ is well defined. Then $1 + M = f(b^n) = tb^n + M + x$, $t \in R, x \in X$. $1 - tb^n + M = x \in R \mathcal{M} \cap X = 0$, $1 - tb^n \in M$, $tb^n \in M$ and hence $1 \in M$, which is contradiction. This shows that $L \oplus K = R$. Then $b^nR + r(b^n)eR$, $e^2 = e \in R$, so $b^n = b^2eb^{n-2} = b^{n-2}a^n b^2$ for some $a^n \in R$, but $b^n \in J(R)$, this implies $b = 0$. This gives $J(R) = 0$. Converse is clear.

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