A Tripled Fixed Point Theorem In Fuzzy Metric Space
Satisfying $\phi$-Contractive Condition

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Abstract: In this paper we prove a tripled common fixed point theorem for four mappings satisfying $\phi$-contractive condition in fuzzy metric spaces, by using the notions of compatibility and subsequential continuity. The result generalizes and extends several classical and very recent related results in literature.

Keywords: tripled fixed point, fuzzy metric space, Hadzic type $t$-norm, compatible mapping, reciprocal continuity, subsequentially continuity.

I. INTRODUCTION

The concept of fuzzy mathematics commenced with the introduction of the notion of fuzzy sets by Zadeh [18]. Fuzzy set theory has a wide application in applied sciences such as mathematical programming, model theory, engineering sciences, image processing, and control theory. Kramosil and Michalek [15] gave the concept of fuzzy metric space in 1975. In 1994, George and Veeramani [1] defined a Hausdorff topology on fuzzy metric space by the notion of fuzzy metric space. Afterwards, the completeness of the fuzzy metric space is defined by Grabiec [20] and he also extended the Banach contraction principle to fuzzy metric spaces. Mishra et al. [31] gave the concept of compatible mappings (introduced by Jungck [13] in metric spaces) which is extended to fuzzy metric spaces and proved common fixed point theorems using continuity of at least one of the mappings. Further, Singh and Jain [6] gave the concept of compatibility by using the concept of weakly compatible mappings in fuzzy metric spaces. The notions of subcompatibility and subsequential continuity to fuzzy metric spaces are extended by Gopal and Imdad [9]. Recently various fixed point theorems applying more general contractive conditions are proved by several authors (e.g., [2,7,8,10,11,12,19,21,27,28]). Bhaskar and Lakshmikantham [32] and Ciric [35] introduced the notion of coupled fixed points and proved some coupled fixed point results in partially ordered metric spaces. Sedghi et al. [31] proved common coupled fixed point theorems in fuzzy metric spaces for commuting mappings in 2010. Hu [34] proved a coupled fixed point theorem for compatible mappings under $\phi$-contractive conditions in fuzzy metric spaces with continuous $t$-norm of H-type motivated by the results of Fang [17] and generalized the result of Sedghi et al. [30].

Pant et al. [3] proved coupled common fixed point theorems for two pairs of mappings satisfying a general contractive condition in fuzzy metric spaces, by using the notions of compatibility and subsequential continuity (alternately subcompatibility and reciprocal continuity).

Inspired by the work of Pant et al. [3] we prove tripled common fixed point theorem for four mappings satisfying $\phi$-contractive condition in fuzzy metric spaces, by using the notions of compatibility and subsequential continuity.

II. PRELIMINARIES

**Definition 2.1** [18]. Let $X$ be any set. A fuzzy set in $X$ is a function with domain $X$ and values $[0,1]$.

**Definition 2.2** [5]. A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a continuous $t$-norm if $*$ satisfies the following conditions:
(a) $*$ is commutative and associative;
(b) $*$ is continuous;
(c) $a \ast 1 = a$ for all $a \in [0,1]$;
(d) $a \ast b \leq c \ast d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

**Definition 2.3**[27]. One says that a $t$-norm $*$ is of H-type if the family $\{*^n\}$ of its iterates is equicontinuous at $x = 1$; that is, for any $\lambda \in (0,1)$, there exists $\delta(\lambda) \in (0,1)$ such that $x > 1 - \delta$ implies $*^n(x) > 1 - \lambda$, for all $n \in N$.

The $t$-norm $*_m = \min\{a, b\}$ for all $a, b \in [0,1]$ is an example of $t$-norm of H-type, but there are some other $t$-norms * of H-type (see [23]).

**Definition 2.4**[15]. A 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if $X$ is an arbitrary nonempty set, $*$ is a continuous $t$-norm, and $M$ is a fuzzy set in $X^2 \times (0, +\infty)$ satisfying the following conditions, for each $x, y, z \in X$ and $t, s > 0$:

(a) $M(x, y, t) > 0$;
(b) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;
(c) $M(x, y, t) = M(y, x, t)$;
(d) $M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s)$;
(e) $M(x, y, \cdot) : (0, \infty) \to [0,1]$ is continuous.

**Example 6**[30]. Let $(X, d)$ be a metric space and $\psi$ be an increasing and continuous function from $R^+ \text{ into } (0,1)$ such that $\log_{s=\infty} \psi(s) = 1$. Four typical examples of these functions are $\psi(s) = s/(s + 1), \psi(s) = sin(\pi s/(2s + 1)), \psi(s) = 1 - e^{-s}$, and $\psi(s) = e^{-1/s}$. Let $a \ast b = ab$ for all $a, b \in [0,1]$, and, for each $x, y \in X$ and $t > 0$, define $M(x, y, t) = [\psi(t)]^{\phi(x,y)}$.

It is easy to see that $(X, M, *)$ is a fuzzy metric space.

**Definition 2.5**[34]. Define $\Phi = \{\phi : R^+ \to R^+\}$ such that $\phi \in \Phi$ satisfies the following conditions:

$\phi - 1 \phi$ is nondecreasing;
$\phi - 2 \phi$ is upper semicontinuous from the right;
$\phi - 3 \sum_{n=0}^\infty \phi^n(s) < +\infty$ for all $s > 0$, where $\phi^{n+1}(s) = \phi(\phi^n(s)), n \in N$.

Clearly if $\phi \in \Phi$, then $\phi(s) < s$ for all $s > 0$.

**Lemma 2.6**[17]. Let $(X, M, *)$ be a fuzzy metric space, where $*$ is a continuous $t$-norm of H-type. If there exists $\phi \in \Phi$ such that $M(x, x, \phi(t)) \geq M(x, y, t)$ for all $t > 0$, then $x = y$.

**Definition 2.7**[4]. An element $(x, y, z) \in X \times X \times X$ is called a tripled fixed point of $f : X \times X \times X \to X$ if $f(x, y, z) = x, f(y, z, x) = y$ and $f(z, x, y) = z$.

**Definition 2.8**[14]. An element $(x, y, z) \in X \times X \times X$ is called a tripled coincidence point of the mappings $f : X \times X \times X \to X$ and $g : X \to X$ if $f(x, y, z) = g(x), f(y, z, x) = g(y), f(z, x, y) = g(z)$.

Moreover, $(x, y, z)$ is called a common tripled fixed point of $f$ and $g$ if $f(x, y, z) = g(x) = x, f(y, z, x) = g(y) = y$ and $f(z, x, y) = g(z) = z$.

**Definition 2.10**[33]. The mappings $f : X \times X \times X \to X$ and $g : X \to X$ are called compatible if

$\lim_{n \to \infty} M(\phi(x_n, y_n, z_n), \phi(gx_n, gy_n, gz_n), t) = 1$,
$\lim_{n \to \infty} M(\phi(y_n, z_n, x_n), \phi(gy_n, gz_n, gx_n), t) = 1$,
$\lim_{n \to \infty} M(\phi(z_n, x_n, y_n), \phi(gz_n, gx_n, gy_n), t) = 1$.
for all $t > 0$, whenever $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are sequences in $X$ such that
\[\lim_{n \to \infty} f(x_n, y_n, z_n) = \lim_{n \to \infty} g(x_n) = \alpha,\]
\[\lim_{n \to \infty} f(y_n, z_n, x_n) = \lim_{n \to \infty} g(y_n) = \beta,\]
\[\lim_{n \to \infty} f(z_n, x_n, y_n) = \lim_{n \to \infty} g(z_n) = \gamma\] ..................(7)
for some $\alpha, \beta, \gamma \in X$.

**Definition 2.11[3].** The mappings $f : X \times X \times X \to X$ and $g : X \to X$ are called reciprocally continuous if for sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in $X$
\[\lim_{n \to \infty} f(gx_n, gy_n, gz_n) = f(\alpha, \beta, \gamma),\]
\[\lim_{n \to \infty} gf(x_n, y_n, z_n) = ga,\]
\[\lim_{n \to \infty} f(gy_n, gz_n, gx_n) = f(\beta, \gamma, \alpha),\]
\[\lim_{n \to \infty} gf(y_n, z_n, x_n) = g\beta,\]
\[\lim_{n \to \infty} f(gz_n, gx_n, gy_n) = f(\gamma, \alpha, \beta),\]
\[\lim_{n \to \infty} gf(z_n, x_n, y_n) = g\gamma\] ..................(8)
Whenever
\[\lim_{n \to \infty} f(x_n, y_n, z_n) = \lim_{n \to \infty} g(x_n) = \alpha,\]
\[\lim_{n \to \infty} f(y_n, z_n, x_n) = \lim_{n \to \infty} g(y_n) = \beta,\]
\[\lim_{n \to \infty} f(z_n, x_n, y_n) = \lim_{n \to \infty} g(z_n) = \gamma,\] ..................(9)

For some $\alpha, \beta, \gamma \in X$
If two self-mappings are continuous, then they are obviously reciprocally continuous, but the converse is not true.
Moreover, in the setting of common fixed point theorems for compatible pairs of self mappings satisfying contractive conditions, continuity of one of the mappings implies their reciprocal continuity but not conversely (see [25]).

**Definition 2.12[3].** The mappings $f : X \times X \times X \to X$ and $g : X \to X$ are called subsequentially continuous if and only if there exist sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in $X$ such that
\[\lim_{n \to \infty} f(x_n, y_n, z_n) = \lim_{n \to \infty} g(x_n) = \alpha,\]
\[\lim_{n \to \infty} f(y_n, z_n, x_n) = \lim_{n \to \infty} g(y_n) = \beta,\]
\[\lim_{n \to \infty} f(z_n, x_n, y_n) = \lim_{n \to \infty} g(z_n) = \gamma,\] ..................(10)
For some $\alpha, \beta, \gamma \in X$

**Definition 2.13[3].** The mappings $f : X \times X \times X \to X$ and $g : X \to X$ are called subcompatible if and only if there exist sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in $X$ such that
\[\lim_{n \to \infty} f(x_n, y_n, z_n) = \lim_{n \to \infty} g(x_n) = \alpha,\]
\[\lim_{n \to \infty} f(y_n, z_n, x_n) = \lim_{n \to \infty} g(y_n) = \beta,\]
\[\lim_{n \to \infty} f(z_n, x_n, y_n) = \lim_{n \to \infty} g(z_n) = \gamma,\] ..................(12)
For some $\alpha, \beta, \gamma \in X$

\[\lim_{n \to \infty} M(gf(x_n, y_n, z_n), f(gx_n, gy_n, gz_n), t) = 1,\]
\[\lim_{n \to \infty} M(gf(y_n, z_n, x_n), f(gy_n, gz_n, gx_n), t) = 1,\]
\[\lim_{n \to \infty} M(gf(z_n, x_n, y_n), f(gz_n, gx_n, gy_n), t) = 1\] ..................(13).
Theorem 3.1. Let \((X, M, *)\) be a fuzzy metric space, where * is a continuous t-norm of H-type such that \(M(x, y, t) \to 1\) as \(t \to \infty\) for all \(x, y \in X\). Let \(A, B : X \times X \times X \to X\) and \(P, Q : X \to X\) be four mappings such that

(a) The pairs \((A, P)\) and \((B, Q)\) are compatible;
(b) The pairs \((A, P)\) and \((B, Q)\) are subsequentially continuous;
(c) There exist \(\Phi \in \Phi\) such that

\[
M(A(x, y, z), B(u, v, w), \phi(t)) \geq M(Px, Qu, t) * M(Py, Qv, t) * M(Pz, Qw, t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldot
Hence \((\alpha, \beta, \gamma) \in X \times X \times X\) is a tripled coincidence point of the pair \((A, P)\) and \((\alpha', \beta', \gamma') \in X \times X \times X\) is a tripled coincidence point of the pair \((B, Q)\).

Now we assume that \((\alpha, \beta, \gamma) = (\alpha', \beta', \gamma')\) i.e. \(\alpha = \alpha', \beta = \beta'\) and \(\gamma = \gamma'.\) Since \(*\) is a t-norm of H-type, then for any \(\lambda > 0, \exists \mu > 0\) such that

\[
(1 - \mu) * (1 - \mu) * \ldots \ldots * p \text{ times} \geq 1 - \lambda
\]

\[\quad \text{..................(19)}\]

For all \(p \in \mathbb{N}\).

Since \(M(x, y, z)\) is continuous and \(\lim_{n \to \infty} M(x, y, t) = 1\) for all \(x, y \in X\) there exist \(t_0 \geq 0\) such that \((\alpha, \alpha', t_0) \geq 1 - \mu, \alpha, \beta, t_0 \geq 1 - \mu, M(y, y', t_0) \geq 1 - \mu\).

On the other hand, since \(\phi \in \Phi\) by condition \((\phi - 3)\) we have \(\sum_{n=1}^{\infty} \phi^n(t_0) < \infty\). Then for any \(t > 0, \exists n_0 \in \mathbb{N}\) such that \(t > \sum_{n=n_0}^{\infty} \phi^n(t_0)\). By inequality (14) with \(x = x_n, y = y_n, z = z_n, u = x_n', v = y_n', w = z_n'. \) We have

\[
M \left( A(x_n, y_n, z_n), B(x'_n, y'_n, z'_n), \phi(t_0) \right) \geq M(Px_n, Qx'_n, t_0) \quad M(Py_n, Qy'_n, t_0) \quad M(Pz_n, Qz'_n, t_0)
\]

Letting \(n \to \infty\) we get

\[
M(\alpha, \alpha', \phi(t_0)) \geq M(\alpha, \alpha', t_0) * M(\beta, \beta', t_0) * M(y, y', t_0)
\]

\[\quad \text{..................(20)}\]

Again using inequality (14) with \(x = y_n, y = y_n, z = x_n, u = y'_n, v = z'_n, w = x'_n. \) We have

\[
M \left( A(y_n, z_n, x_n), B(y'_n, z'_n, x'_n), \phi(t_0) \right) \geq M(Py_n, Qy'_n, t_0) * M(Pz_n, Qz'_n, t_0) * M(Px_n, Qx'_n, t_0)
\]

Letting \(n \to \infty\) we get

\[
M(\beta, \beta', \phi(t_0)) \geq M(\beta, \beta', t_0) * M(y, y', t_0) * M(\alpha, \alpha', t_0)
\]

\[\quad \text{...............(21)}\]

Again using inequality (14) with \(x = z_n, y = x_n, z = y_n, u = x'_n, v = x'_n, w = y'_n\)

\[
M \left( A(z_n, x_n, y_n), B(x'_n, y'_n, z'_n), \phi(t_0) \right) \geq M(Pz_n, Qz'_n, t_0) * M(Px_n, Qx'_n, t_0) * M(Py_n, Qy'_n, t_0)
\]

\[
M(\gamma, \gamma', \phi(t_0)) \geq M(\gamma, \gamma', t_0) * M(\alpha, \alpha', t_0) * M(\beta, \beta', t_0)
\]

\[\quad \text{...............(22)}\]

From (20),(21) and (22) We obtain

\[
M(\alpha, \alpha', \phi(t_0)) * M(\beta, \beta', \phi(t_0)) * M(\gamma, \gamma', \phi(t_0)) \geq M(\alpha, \alpha', t_0)^3 * M(\beta, \beta', t_0)^3 * M(\gamma, \gamma', t_0)^3
\]

Similarly we have
\[ M(\alpha, \alpha', \phi(t_0)) \geq M(\alpha, \alpha', \phi(t_0)) \cdot M(\beta, \beta', \phi(t_0)) \cdot M(\gamma, \gamma', \phi(t_0)) \]

\[ \geq M(\alpha, \alpha', t_0) \cdot M(\beta, \beta', t_0) \cdot M(\gamma, \gamma', t_0) \cdot M(\beta, \beta', t_0) \cdot M(\gamma, \gamma', t_0) \cdot \]

\[ \geq M(\alpha, \alpha', t_0)^3 \cdot M(\beta, \beta', t_0)^3 \cdot M(\gamma, \gamma', t_0)^3 \] \[ \quad \text{..........(23)} \]

\[ M(\beta, \phi(t_0)) \geq M(\beta, \phi(t_0)) \cdot M(\gamma, \phi(t_0)) \cdot M(\alpha, \phi(t_0)) \]

\[ \geq M(\beta, \beta', t_0) \cdot M(\gamma, \gamma', t_0) \cdot M(\alpha, \alpha', t_0) \cdot M(\beta, \beta', t_0) \cdot M(\gamma, \gamma', t_0) \cdot \]

\[ \geq M(\beta, \beta', t_0)^3 \cdot M(\gamma, \gamma', t_0)^3 \cdot M(\alpha, \alpha', t_0)^3 \] \[ \quad \text{..........(24)} \]

\[ M(\gamma, \phi(t_0)) \geq M(\gamma, \phi(t_0)) \cdot M(\alpha, \phi(t_0)) \cdot M(\beta, \phi(t_0)) \]

\[ \geq M(\gamma, \gamma', t_0) \cdot M(\alpha, \alpha', t_0) \cdot M(\beta, \beta', t_0) \cdot M(\gamma, \gamma', t_0) \cdot \]

\[ \geq M(\gamma, \gamma', t_0)^3 \cdot M(\alpha, \alpha', t_0)^3 \cdot M(\beta, \beta', t_0)^3 \] \[ \quad \text{..........(25)} \]

From (23), (24) and (25)

\[ M(\alpha, \alpha', \phi^2(t_0)) \cdot M(\beta, \beta', \phi^2(t_0)) \cdot M(\gamma, \phi^2(t_0)) \]

\[ \geq M(\alpha, \alpha', t_0)^{32} \cdot M(\beta, \beta', t_0)^{32} \cdot M(\gamma, \gamma', t_0)^{32} \]

\[ \quad \text{..........(26)} \]

In general for all \( n \in \mathbb{N} \), we have

\[ M(\alpha, \alpha', \phi^n(t_0)) \cdot M(\beta, \beta', \phi^n(t_0)) \cdot M(\gamma, \gamma', \phi^n(t_0)) \]

\[ \geq \{M(\alpha, \alpha', \phi^{n-1}(t_0))\}^3 \cdot \{M(\beta, \beta', \phi^{n-1}(t_0))\}^3 \cdot \{M(\gamma, \gamma', \phi^{n-1}(t_0))\}^3 \]

\[ \geq \{M(\alpha, \alpha', t_0)\}^{3n} \cdot \{M(\beta, \beta', t_0)\}^{3n} \cdot \{M(\gamma, \gamma', t_0)\}^{3n} \]

\[ \quad \text{..........(27)} \]

Then we have

\[ M(\alpha, \alpha', t) \cdot M(\beta, \beta', t) \cdot M(\gamma, \gamma', t) \]

\[ \geq M(\alpha, \alpha', \sum_{\mu=n_0}^{\infty} \phi^\mu(t_0)) \cdot M(\beta, \beta', \sum_{\mu=n_0}^{\infty} \phi^\mu(t_0)) \cdot M(\gamma, \gamma', \sum_{\mu=n_0}^{\infty} \phi^\mu(t_0)) \]

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\[ \geq M(\alpha, \alpha', \phi^n_0(t_0)) \cdot M(\beta, \beta', \phi^n_0(t_0)) \cdot M(\gamma, \gamma', \phi^n_0(t_0)) \]
\[ \geq [M(\alpha, \alpha', t_0)]^{3^n_0} \cdot [M(\beta, \beta', t_0)]^{3^n_0} \cdot [M(\gamma, \gamma', t_0)]^{3^n_0} \]
\[ \geq (1 - \mu) \cdot (1 - \mu) \cdot \ldots \cdot 3^{n_0+1} \text{times} \]
\[ \geq 1 - \lambda \]

So for any \( \lambda > 0 \), we have

\[ M(\alpha, \alpha', t) \cdot M(\beta, \beta', t) \cdot M(\gamma, \gamma', t) \geq 1 - \lambda \]  
\[ \ldots \ldots (28) \]

for all \( t > 0 \), and so \( \alpha = \alpha', \beta = \beta' \) and \( \gamma = \gamma' \). Therefore, we have

\[ A(\alpha, \beta, \gamma) = P\alpha \cdot A(\beta, \gamma, \alpha) = P\beta \] and \( A(\gamma, \alpha, \beta) = P\gamma \)

\[ B(\alpha, \beta, \gamma) = Q\alpha \cdot B(\beta, \gamma, \alpha) = Q\beta \] and \( B(\gamma, \alpha, \beta) = Q\gamma \)  
\[ \ldots \ldots (29) \]

Next we show that \( P\alpha = Q\alpha, P\beta = Q\beta \) and \( P\gamma = Q\gamma \).

Since \( \ast \) is a t-norm of H-type, then for any \( \lambda > 0, \exists \mu > 0 \) such that

\[ (1 - \mu) \ast (1 - \mu) \ast \ldots \ast p \text{ times} \geq 1 - \lambda \]  
\[ \ldots \ldots (30) \]

For all \( \epsilon \in \mathbb{N} \).

Since \( M(x, y, \ast) \) is continuous and \( \lim_{n \to \infty} M(x, y, t) = 1 \) for all \( x, y \in X \) there exist \( t_0 > 0 \) such that

\[ M(\alpha, \beta, \gamma) \geq M(P\alpha, Q\alpha, t_0) \cdot M(P\beta, Q\beta, t_0) \cdot M(P\gamma, Q\gamma, t_0) \]

Letting \( n \to \infty \) we get

\[ M(P\alpha, Q\alpha, \phi(t_0)) \geq M(P\alpha, Q\alpha, t_0) \cdot M(P\beta, Q\beta, t_0) \cdot M(P\gamma, Q\gamma, t_0) \]  
\[ \ldots \ldots (31) \]

Similarly on using \( x = \beta, y = \gamma, z = \alpha \). we can obtain

\[ M(P\beta, Q\beta, \phi(t_0)) \geq M(P\beta, Q\beta, t_0) \cdot M(P\gamma, Q\gamma, t_0) \cdot M(P\alpha, Q\alpha, t_0) \]  
\[ \ldots \ldots (32) \]

On using \( x = \gamma, y = \alpha, z = \beta \). we can get

\[ M(P\gamma, Q\gamma, \phi(t_0)) \geq M(P\gamma, Q\gamma, t_0) \ast M(P\alpha, Q\alpha, t_0) \ast M(P\beta, Q\beta, t_0) \]  
\[ \ldots \ldots (33) \]

From (31), (32) and (33) we obtain

\[ M(P\alpha, Q\alpha, \phi(t_0)) \ast M(P\beta, Q\beta, \phi(t_0)) \ast M(P\gamma, Q\gamma, \phi(t_0)) \]
\[ \geq M(P\alpha, Q\alpha, t_0) \cdot M(P\beta, Q\beta, t_0) \cdot M(P\gamma, Q\gamma, t_0) \]
\[ \geq M(P\alpha, Q\alpha, t_0)^3 \cdot M(P\beta, Q\beta, t_0)^3 \cdot M(P\gamma, Q\gamma, t_0)^3 \]
In general for all \( n \in \mathbb{N} \), we have
\[
M(Pa, Q\alpha, \phi^n(t_0)) \ast M(P\beta, Q\beta, \phi^n(t_0)) \ast M(Py, Qy, \phi^n(t_0))
\]
\[
\geq [M(Pa, Q\alpha, \phi^{n-1}(t_0))]^3 \ast [M(P\beta, Q\beta, \phi^{n-1}(t_0))]^3 \ast [M(Py, Qy, \phi^{n-1}(t_0))]^3
\]
\[
\geq [M(Pa, Q\alpha, t_0)]^3 \ast [M(P\beta, Q\beta, t_0)]^3 \ast [M(Py, Qy, t_0)]^3
\]  
(34)

Then we have
\[
M(Pa, Q\alpha, t) \ast M(P\beta, Q\beta, t) \ast M(Py, Qy, t) \geq M(Pa, Q\alpha, \sum_{n=0}^{\infty} \phi^n(t_0)) \ast M(P\beta, Q\beta, \sum_{n=0}^{\infty} \phi^n(t_0)) \ast M(Py, Qy, \phi^n(t_0)) \geq [M(Pa, Q\alpha, t_0)]^{3n} \ast [M(P\beta, Q\beta, t_0)]^{3n} \ast [M(Py, Qy, t_0)]^{3n}
\]
\[
\geq (1 - \mu) \ast (1 - \mu) \ast \ldots \ast 3^{n+1} \text{ times}
\]
\[
\geq 1 - \lambda
\]
(35)

So for any \( \lambda > 0 \), we have
\[
M(Pa, Q\alpha, t) \ast M(P\beta, Q\beta, t) \ast M(Py, Qy, t) \geq 1 - \lambda
\]  
(36)

for all \( t > 0 \), and so \( Pa = Q\alpha, P\beta = Q\beta \) and \( Py = Qy \). Therefore we have
\[
Pa = Q\alpha = A(\alpha, \beta, y) = B(\alpha, \beta, y).
\]
\[
P\beta = Q\beta = A(\beta, y, \alpha) = B(\beta, y, \alpha)
\]
\[
Py = Qy = A(y, \alpha, \beta) = B(y, \alpha, \beta)
\]
(37)

Now we show that \( Sa = \alpha \) and \( S\beta = \beta \). Since \( * \) is a \( t \)-norm of \( H \)-type, for any \( \lambda > 0 \), there exists an \( \mu > 0 \) such that \( (1 - \mu) \ast (1 - \mu) \ast \ldots \ast p \text{ times} \geq 1 - \lambda \)

for all \( p \in \mathbb{N} \).

Since \( M(x, y, \cdot) \) is continuous and \( \lim_{n \to \infty} M(x, y, t) = 1 \) for all \( x, y \in X \) there exist \( t_0 > 0 \) such that \( (Pa, \alpha, t_0) \geq 1 - \mu \)
\[
M(P\beta, \beta, t_0) \geq 1 - \mu, M(Py, y, t_0) \geq 1 - \mu
\]

On the other hand, since \( \phi \in \Phi \) by condition (\( \phi - 3 \)) we have \( \sum_{n=1}^{\infty} \phi^n(t_0) < \infty \). Then for any \( t > 0 \), \( \exists \eta_0 \in \mathbb{N} \) such that \( t > \sum_{n=0}^{\eta_0} \phi^n(t_0) \). By inequality (14) with \( x = \alpha, y = \beta, z = y, u = x, v = y, w = z \). We have
\[
M(A(\alpha, \beta, y, B(\alpha, \beta, y, z), \phi(t_0))) \geq M(Pa, Qx, t_0) \ast M(P\beta, Qy, t_0) \ast M(Py, Qz, t_0)
\]

Letting \( n \to \infty \) we get
\[
M(Pa, \alpha, \phi(t_0)) \geq M(P\alpha, \alpha, t_0) \ast M(P\beta, \beta, t_0) \ast M(Py, y, t_0)
\]
(38)

Similarly on using \( x = \beta, y = \gamma, z = \alpha, u = y, v = z, w = x \) we have
\[
M(P\beta, \beta, \phi(t_0)) \geq M(P\beta, \beta, t_0) \ast M(Py, y, t_0) \ast M(Pa, \alpha, t_0)
\]
(39)

On using \( x = \gamma, y = \alpha, z = \beta, u = z, v = x, w = y \) we have
We have \( M(Py, y, \phi(t_o)) \geq M(Py, y, t_o) * M(\alpha, \alpha, t_o) * M(P\beta, \beta, t_o) \)  

\[ \ldots \ldots .(40) \]

Consequently from (38), (39) and (40)

\[ M(\alpha, \phi(t_o)) * M(\beta, \phi(t_o)) * M(Py, \phi(t_o)) \]

\[ \geq M(\alpha, \phi(t_o))^3 * M(\beta, \phi(t_o))^3 * M(Py, \phi(t_o))^3 \]

In general for all \( n \in \mathbb{N} \), we have \( M(\alpha, \phi^n(t_o)) * M(\beta, \phi^n(t_o)) * M(Py, \phi^n(t_o)) \)

\[ \geq \left[M(\alpha, \phi^{n-1}(t_o))\right]^3 * \left[M(\beta, \phi^{n-1}(t_o))\right]^3 * \left[M(Py, \phi^{n-1}(t_o))\right]^3 \]

\[ \geq \left[M(\alpha, t_o)\right]^3n * \left[M(\beta, t_o)\right]^3n * \left[M(Py, t_o)\right]^3n \]

\[ \ldots \ldots .(41) \]

Then we have

\[ M(\alpha, \phi, t) * M(\beta, \phi, t) * M(Py, \phi, t) \]

\[ \geq M(\alpha, \sum_{p=0}^{\infty} \phi(t_o)) * M(\beta, \sum_{p=0}^{\infty} \phi(t_o)) * M(Py, \sum_{p=0}^{\infty} \phi(t_o)) \]

\[ \geq M(\alpha, \phi^n(t_o)) * M(\beta, \phi^n(t_o)) * M(Py, \phi^n(t_o)) \]

\[ \geq \left[M(\alpha, \phi(t_o))\right]^3n * \left[M(\beta, \phi(t_o))\right]^3n * \left[M(Py, \phi(t_o))\right]^3n \]

\[ \geq (1 - \mu) * (1 - \mu) * \ldots \ldots 3^n \text{ times} \]

\[ \geq 1 - \lambda \]

So for any \( \lambda > 0 \), we have \( M(\alpha, \phi, t) * M(\beta, \phi, t) * M(Py, \phi, t) \geq 1 - \lambda \)  

\[ \ldots \ldots .(42) \]

for all \( t > 0 \), and so \( P\alpha = \alpha, P\beta = \beta \) and \( Py = y \). Thus

\[ \alpha = P\alpha = Q\alpha = A(\alpha, \beta, y) = B(\alpha, \beta, y), \]

\[ \beta = P\beta = Q\beta = A(\beta, y, \alpha) = B(\beta, y, \alpha) \]

\[ \gamma = Py = Qy = A(y, \alpha, \beta) = B(y, \alpha, \beta) \]

Finally we assume that \( \alpha = \beta = y \). Since * is a \( t \)-norm of \( H \)-type, for any \( \lambda > 0 \), there exists an \( \mu > 0 \) such that \( (1 - \mu) * (1 - \mu) * \ldots \ldots p \text{ times} \geq 1 - \lambda \)

For all \( p \in \mathbb{N} \).

Since \( M(x, y, \cdot) \) is continuous and \( \lim_{n \to \infty} M(x, y, t) = 1 \) for all \( x, y \in X \) there exist \( t_o > 0 \) such that \( \alpha, \beta, t_o \geq 1 - \mu \), \( M(\beta, \gamma, t_o) \geq 1 - \mu \), \( M(\alpha, \gamma, t_o) \geq 1 - \mu \).

On using \( x = w = x_n, y = u = y_n, z = v = z_n \) in (14), we have

\[ M(A(x_n, y_n, z_n), B(y_n, z_n, x_n), \phi(t_o)) \]
Letting $n \to \infty$ we have

$$M(\alpha, \beta, \phi(t_0)) \geq M(\alpha, \beta, 0) \ast M(\beta, \gamma, 0) \ast M(\gamma, \alpha, 0)$$  \hspace{1cm}  \text{(43)}$$

Again on using $x = y = u = z_n, z = v = x_n$ in (14) we have

$$M(A(y_n, z_n, x_n), B(z_n, y_n, x_n) \phi(t_0)) \geq M(P y_n, Q z_n, t_0) \ast M(P z_n, Q x_n, t_0) \ast M(P x_n, Q y_n, t_0)$$

Letting $n \to \infty$ we have

$$M(\beta, \gamma, \phi(t_0)) \geq M(\gamma, \alpha, t_0) \ast M(\alpha, \beta, t_0) \ast M(\alpha, \beta, 0)$$  \hspace{1cm}  \text{(44)}$$

Again on using $x = w = z_n, y = u = x_n, z = v = y_n$ in (14) we have $M(A(z_n, y_n, x_n), B(x_n, y_n, z_n) \phi(t_0)) \geq M(P z_n, Q x_n, t_0) \ast M(P x_n, Q y_n, t_0) \ast M(P y_n, Q z_n, t_0)$

Letting $n \to \infty$ we have

$$M(\gamma, \alpha, \phi(t_0)) \geq M(\gamma, \alpha, t_0) \ast M(\alpha, \beta, t_0) \ast M(\alpha, \beta, 0) \ast M(\beta, \gamma, t_0)$$  \hspace{1cm}  \text{(45)}$$

From (43),(44),(45)

$$M(\alpha, \beta, \phi(t_0)) \ast M(\beta, \gamma, \phi(t_0)) \ast M(\gamma, \alpha, \phi(t_0)) \geq M(\alpha, \beta, t_0)^3 \ast M(\beta, \gamma, t_0)^3 \ast M(\gamma, \alpha, t_0)^3$$

In general for all $n \in \mathbb{N}$, we have

$$M(\alpha, \beta, \phi^n(t_0)) \ast M(\beta, \gamma, \phi^n(t_0)) \ast M(\gamma, \alpha, \phi^n(t_0))$$

$$\geq [M(\alpha, \beta, \phi^{n-1}(t_0))]^3 \ast [M(\beta, \gamma, \phi^{n-1}(t_0))]^3 \ast [M(\gamma, \alpha, \phi^{n-1}(t_0))]^3$$

$$\geq [M(\alpha, \beta, t_0)]^{3n} \ast [M(\beta, \gamma, t_0)]^{3n} \ast [M(\gamma, \alpha, t_0)]^{3n}$$  \hspace{1cm}  \text{(46)}$$

Then we have

$$M(\alpha, \beta, t) \ast M(\beta, \gamma, t) \ast M(\gamma, \alpha, t)$$

$$\geq M(\alpha, \beta, \sum_{p=n_0}^{\infty} \phi^p(t_0)) \ast M(\beta, \gamma, \sum_{p=n_0}^{\infty} \phi^p(t_0)) \ast M(\gamma, \alpha, \sum_{p=n_0}^{\infty} \phi^p(t_0))$$

$$\geq M(\alpha, \beta, \phi^{n_0}(t_0)) \ast M(\beta, \gamma, \phi^{n_0}(t_0)) \ast M(\gamma, \alpha, \phi^{n_0}(t_0))$$

$$\geq [M(\alpha, \beta, t_0)]^{3n_0} \ast [M(\beta, \gamma, t_0)]^{3n_0} \ast [M(\gamma, \alpha, t_0)]^{3n_0}$$

$$\geq (1 - \mu) \ast (1 - \mu) \ast \cdots \cdots 3^{n_0+1} \text{ times}$$

$$\geq 1 - \lambda$$

Which implies that $\alpha = \beta = \gamma$. 

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Corollary 3.2. Let \((X,M,\ast)\) be a fuzzy metric space, where \(\ast\) is a continuous \(t\)-norm of \(H\)-type such that \(M(x,y,t) \to 1\) as \(t \to \infty\) for all \(x,y \in X\). Let \(A:X \times X \times X \to X\) and \(P:X \to X\) be compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous) mappings such that

\[
M(A(x,y,z), A(u,v,w), \phi(t)) \geq M(Px, Pu, t) \ast M(Py, Pv, t) \ast M(Pz, Pw, t) \quad \ldots (47)
\]

for all \(x,y,z,u,v,w \in X\), \(\phi \in \Phi\) and \(t > 0\).

Then exists a unique point \(a \in X\) such that \(a = Pa = A(a,a,a).

Example: 3.3 Let \(X = [0,\infty), a \ast b = ab\) for all \(a, b \in [0,1]\) and \(\psi(s) = \frac{s}{s+1}\) for all \(s \in R^+\). Then \((X, M, \ast)\) is a fuzzy metric space, where \(M(x,y,t) = |\psi(s)|^{x-y}\)

for all \(x,y,z,u,v,w \in X\), and \(t > 0\). Let \(\phi(s) = s/2\) and let the mappings \(A:X \times X \times X \to X\) and \(P:X \to X\) be defined as

\[
A(x,y,z) = \begin{cases} 
2x + 2y + 2z - 5, & \text{if } x,y,z \in [1,\infty) \\
\frac{x + y + z}{6}, & \text{otherwise}
\end{cases}
\]

\[
P(x) = \begin{cases} 
3x - 2, & \text{if } x \in [1,\infty) \\
\frac{x}{6}, & \text{otherwise}
\end{cases}
\]

In view of definition 2.10, to prove compatibility we have only to consider sequences converging to zero from the right. In such case we have

\[
\lim_{n \to \infty} A(x_n, y_n, z_n) = 0 = \lim_{n \to \infty} P(x_n)
\]

\[
\lim_{n \to \infty} A(y_n, z_n, x_n) = 0 = \lim_{n \to \infty} P(y_n),
\]

\[
\lim_{n \to \infty} A(z_n, x_n, y_n) = 0 = \lim_{n \to \infty} P(z_n).
\]

Next we get

\[
\lim_{n \to \infty} A(Px_n, Py_n, Pz_n) = 0 = A(0,0,0)
\]

\[
\lim_{n \to \infty} PA(x_n, y_n, z_n) = 0 = P(0),
\]

\[
\lim_{n \to \infty} A(Py_n, Pz_n, Px_n) = 0 = A(0,0,0),
\]

\[
\lim_{n \to \infty} PA(z_n, x_n, y_n) = 0 = P(0),
\]

\[
\lim_{n \to \infty} A(Pz_n, Px_n, Py_n) = 0 = A(0,0,0)
\]

\[
\lim_{n \to \infty} PA(y_n, z_n, x_n) = 0 = P(0).
\]
Consequently
\[ \lim_{n \to \infty} M(A(Px_n, Py_n, Pz_n), PA(x_n, y_n, z_n), t) = 1 \]
\[ \lim_{n \to \infty} M(A(Py_n, Pz_n, Px_n), PA(y_n, z_n, x_n), t) = 1 \]
\[ \lim_{n \to \infty} M(A(Pz_n, Px_n, Py_n), PA(z_n, x_n, y_n), t) = 1 \]

For all \( t > 0 \).

On the other hand, to prove subsequential continuity, in view of Definition 2.12, we have only to consider sequences \( \{x_n\}, \{y_n\} \) and \( \{z_n\} \) converging to one from the right. In such case we have
\[ \lim_{n \to \infty} A(x_n, y_n, z_n) = 1 = \lim_{n \to \infty} P(x_n) \]
\[ \lim_{n \to \infty} A(y_n, z_n, x_n) = 1 = \lim_{n \to \infty} P(y_n) \]
\[ \lim_{n \to \infty} A(z_n, x_n, y_n) = 1 = \lim_{n \to \infty} P(z_n) \]

Next we get
\[ \lim_{n \to \infty} A(Px_n, Py_n, Pz_n) = 1 = A(1, 1, 1) \]
\[ \lim_{n \to \infty} PA(x_n, y_n, z_n) = 1 = P(1) \]
\[ \lim_{n \to \infty} A(Py_n, Pz_n, Px_n) = 1 = A(1, 1, 1) \]
\[ \lim_{n \to \infty} PA(y_n, z_n, x_n) = 1 = P(1) \]
\[ \lim_{n \to \infty} A(Pz_n, Px_n, Py_n) = 1 = A(1, 1, 1) \]
\[ \lim_{n \to \infty} PA(z_n, x_n, y_n) = 1 = P(1) \]

Again for the same sequences we have
\[ \lim_{n \to \infty} M(A(Px_n, Py_n, Pz_n), PA(x_n, y_n, z_n), t) = 1 \]
\[ \lim_{n \to \infty} M(A(Py_n, Pz_n, Px_n), PA(y_n, z_n, x_n), t) = 1 \]
\[ \lim_{n \to \infty} M(A(Pz_n, Px_n, Py_n), PA(z_n, x_n, y_n), t) = 1 \]

Thus the mappings \( A \) and \( P \) are compatible as well as subsequentially continuous and reciprocally continuous.

Next, by a routine calculation, one can verify that condition (47) is true. For instance, for all \( t > 0 \) and \( x, y, z, u, v, w \in [0, 1] \), we have
\[
M(A(x, y, z), A(u, v, w), \phi(t)) = M \left( A(x, y, z), A(u, v, w), \frac{t}{2} \right)
= \left[ \frac{x + u + y + v + z + w}{6} \right] \phi \left( \frac{t}{2} \right)
\]

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Therefore, all the conditions of Corollary 3.2 are satisfied. $(0,0)$ and $(1,1,1)$ are common fixed points of the pair $(A, P)$.

REFERENCES