A Fixed Point Theorems for Soft Contractive Mapping by Using Altering Distance Function

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\textbf{ABSTRACT}

In the present paper, we prove some fixed point theorem in complete soft metric spaces by using altering distance function.

\textbf{Keywords:} - Soft metric space, soft contractive mapping, fixed point, altering distance function.

\textbf{Mathematics Subject Classification:} - 47H10, 54H25.

\section{1. INTRODUCTION & PRELIMINARIES}

A new category of contractive fixed point problem was introduced by M. S. Khan, M. Swalech and S. Sessa [9]. In this work, they introduced the concept of altering distance function which is a control function that alters distance between two points in a metric space.

In the year 1999, Molodtsov [13] initiated a novel concept of soft sets theory as a new mathematical tool for dealing with uncertainties. A soft set is a collection of approximate descriptions of an object. Soft systems provide a very general framework with the involvement of parameters. Since soft set theory has a rich potential, applications of soft set theory in other disciplines and real life problems are progressing rapidly.

Maji et al. [10,11] worked on soft set theory and presented an application of soft sets in decision making problems. Chen [3] introduced a new definition of soft set parameterization reduction and a comparison of it with attribute reduction in rough set theory. Many researchers contributed towards many structure on soft set theory. [1, 3].

M. Shabir and M. Naz [14] presented soft topological spaces and they investigated some properties of soft topological spaces. Later, many researches about soft topological spaces were studied in [2,6,7,12,14]. In these studies, the concept of soft point is expressed by different approaches. In the study we use the concept of soft point which was given in [2,4].

\textbf{Definition 2.1}: Let $X$ be an initial universe set and $E$ be a set of parameters. A pair $(F, E)$ is called a soft set over $X$ if and only if $X$ is a mapping from $E$ into the set of all subsets of the set $X$, i.e. $F: E \rightarrow P(X)$, where $P(X)$ is the power set of $X$.

\textbf{Definition 2.2}: The intersection of two soft sets $(F, A)$ and $(G, B)$ over $X$ is the soft set $(H, C)$, where $C = A \cap B$ and $\forall \epsilon \in C, H(\epsilon) = F(\epsilon) \cap G(\epsilon)$. This is denoted by $(F, A) \cap (G, B) = (H, C)$. 

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Definition 2.3: The union of two soft sets \((F, A)\) and \((G, B)\) over \(X\) is the soft set, where \(C = A \cup B\) and \(\forall \varepsilon \in C\),

\[
H(\varepsilon) = \begin{cases} 
F(\varepsilon), & \text{if } \varepsilon \in A - B \\
G(\varepsilon), & \text{if } \varepsilon \in B - A \\
F(\varepsilon) \cup G(\varepsilon), & \varepsilon \in A \cap B 
\end{cases}
\]

This relationship is denoted by \( (F, A) \cup (G, B) = (H, C) \).

Definition 2.4: The soft set \((F, A)\) over \(X\) is said to be a null soft set denoted by \(\Phi\) if for all \(\varepsilon \in A, F(\varepsilon) = \phi\) (null set).

Definition 2.5: A soft set \((F, A)\) over \(X\) is said to be an absolute soft set, if for all \(\varepsilon \in A, F(\varepsilon) = X\).

Definition 2.6: The difference \((H, E)\) of two soft sets \((H, E)\) and \((H, E)\) over \(X\) denoted by \((H, E) \setminus (H, E)\), is defined as \(H(e) = F(e) \setminus G(e)\) for all \(e \in E\).

Definition 2.7: The complement of a soft set \((F, A)\) is denoted by \((F, A)^c\) and is defined by \((F, A)^c = (F^c, A)\) where \(F^c: A \to P(X)\) is mapping given by \(F^c(\alpha) = X - F(\alpha), \forall \alpha \in A\).

Definition 2.8: Let \(\mathcal{R}\) be the set of real numbers and \(B(\mathcal{R})\) be the collection of all nonempty bounded subsets of \(\mathcal{R}\) and \(E\) taken as a set of parameters. Then a mapping \(F: E \to B(\mathcal{R})\) is called a soft real set. It is denoted by \((F, E)\). If specifically \((F, E)\) is a singleton soft set, then identifying \((F, E)\) with the corresponding soft element, it will be called a soft real number and denoted \(\tilde{r}, \tilde{s}, \tilde{t}\) etc.

\(\tilde{0}, \tilde{1}\) are the soft real numbers where \(\tilde{0}(e) = 0, \tilde{1}(e) = 1\) for all \(e \in E\), respectively.

Definition 2.9: For two soft real numbers

(i) \(\tilde{r} \leq \tilde{s}\), if \(\tilde{r}(e) \leq \tilde{s}(e)\), for all \(e \in E\).

(ii) \(\tilde{r} \geq \tilde{s}\), if \(\tilde{r}(e) \geq \tilde{s}(e)\), for all \(e \in E\).

(iii) \(\tilde{r} < \tilde{s}\), if \(\tilde{r}(e) < \tilde{s}(e)\), for all \(e \in E\).

(iv) \(\tilde{r} > \tilde{s}\), if \(\tilde{r}(e) > \tilde{s}(e)\), for all \(e \in E\).
Definition 2.10: A soft set over $X$ is said to be a soft point if there is exactly one $e \in E$, such that $P(e) = \{x\}$ for some $x \in X$ and $P(e') = \emptyset, \forall e' \in E\{e\}$. It will be denoted by $\tilde{x}_e$.

Definition 2.11: Two soft points $\tilde{x}_e, \tilde{y}_e$ are said to be equal if $e = e'$ and $P(e) = P(e')$ i.e. $x = y$. Thus $\tilde{x}_e \neq \tilde{y}_e \iff x \neq y$ or $e \neq e'$.

Definition 2.12: A mapping $\tilde{d}: SP(\tilde{X}) \times SP(\tilde{X}) \to \mathbb{R}(E)^*$, is said to be a soft metric on the soft set $\tilde{X}$ if $d$ satisfies the following conditions:

\begin{align*}
\text{(M1)} & \quad \tilde{d}(\tilde{x}_{e_1}, \tilde{y}_{e_2}) \leq \tilde{0} \text{ for all } \tilde{x}_{e_1}, \tilde{y}_{e_2} \in \tilde{X}, \\
\text{(M2)} & \quad \tilde{d}(\tilde{x}_{e_1}, \tilde{y}_{e_2}) = \tilde{0} \text{ if and only if } \tilde{x}_{e_1} = \tilde{y}_{e_2}, \\
\text{(M3)} & \quad \tilde{d}(\tilde{x}_{e_1}, \tilde{y}_{e_2}) \leq \tilde{d}(\tilde{y}_{e_2}, \tilde{x}_{e_1}) \text{ for all } \tilde{x}_{e_1}, \tilde{y}_{e_2} \in \tilde{X}, \\
\text{(M4)} & \quad \tilde{d}(\tilde{x}_{e_1}, \tilde{z}_{e_3}) \leq \tilde{d}(\tilde{x}_{e_1}, \tilde{y}_{e_2}) + \tilde{d}(\tilde{y}_{e_2}, \tilde{z}_{e_3}) \text{ for all } \tilde{x}_{e_1}, \tilde{y}_{e_2}, \tilde{z}_{e_3} \in \tilde{X}.
\end{align*}

The soft set $\tilde{X}$ with a soft metric $\tilde{d}$ on $\tilde{X}$ is called a soft metric space and denoted by $(\tilde{X}, \tilde{d}, E)$.

Definition 2.13 (Cauchy Sequence): A sequence $\{\tilde{x}_{\lambda,n}\}_n$ of soft points in $(\tilde{X}, \tilde{d}, E)$ is considered as a Cauchy sequence in $\tilde{X}$ if corresponding to every $\tilde{\epsilon} \geq \tilde{0}, \exists m \in N$ such that $\tilde{d}(\tilde{x}_{\lambda,i}, \tilde{x}_{\lambda,j}) \leq \tilde{\epsilon}, \forall i, j \geq m$, i.e. $d(\tilde{x}_{\lambda,i}, \tilde{x}_{\lambda,j}) \to \tilde{0}$ as $i, j \to \infty$.

Definition 2.14 (Soft Complete Metric Space): A soft metric space $(\tilde{X}, \tilde{d}, E)$ is called complete, if every Cauchy Sequence in $\tilde{X}$ converges to some point of $\tilde{X}$.

Definition 2.15: Let $(\tilde{X}, \tilde{d}, E)$ be a soft metric space. A function $(f, \varphi) : (\tilde{X}, \tilde{d}, E) \to (\tilde{X}, \tilde{d}, E)$ is called a soft contractive mapping if there exist a soft real number $\alpha \in R, 0 \leq \alpha < 1$ such that for every point $\tilde{x}_{\lambda}, \tilde{y}_{\mu} \in SP(\tilde{X})$ we have

\[ \tilde{d}((f, \varphi)(\tilde{x}_{\lambda}), (f, \varphi)(\tilde{y}_{\mu})) \leq \alpha \tilde{d}(\tilde{x}_{\lambda}, \tilde{y}_{\mu}) \]

Definition 2.16[9]: The function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:

(i) $\psi$ is continuous and non-decreasing,
(ii) $\psi(t) = 0$ if and only if $t = 0$. 

3. MAIN RESULTS

**Theorem 3.1:** Let \((\bar{X}, \tilde{d}, E)\) be a soft complete metric space. Suppose the soft mapping \((f, \varphi) : (\bar{X}, \tilde{d}, E) \rightarrow (\bar{X}, \tilde{d}, E)\) satisfies the soft contractive condition:

\[
\psi \left[ \tilde{d} \left( (f, \varphi)(\bar{x}_\lambda), (f, \varphi)(\bar{y}_\mu) \right) \right] \leq \psi \left[ \tilde{M}(\bar{x}_\lambda, \bar{y}_\mu) \right] \quad \text{for each} \quad \bar{x}_\lambda, \bar{y}_\mu \in \bar{X}, \bar{x}_\lambda \neq \bar{y}_\mu,
\]

where \(\psi, \varphi\) are altering distance functions, and

\[
\tilde{M}(\bar{x}_\lambda, \bar{y}_\mu) = \alpha \left( \frac{\tilde{d}(\bar{x}_\lambda, (f, \varphi)(\bar{x}_\lambda)) + \tilde{d}(\bar{y}_\mu, (f, \varphi)(\bar{y}_\mu))}{1 + \tilde{d}(\bar{x}_\lambda, (f, \varphi)(\bar{x}_\lambda)) + \tilde{d}(\bar{y}_\mu, (f, \varphi)(\bar{y}_\mu))} \right) + \beta \left( \frac{\tilde{d}(\bar{x}_\lambda, (f, \varphi)(\bar{x}_\lambda)) + \tilde{d}(\bar{y}_\mu, (f, \varphi)(\bar{y}_\mu))}{1 + \tilde{d}(\bar{x}_\lambda, (f, \varphi)(\bar{x}_\lambda)) + \tilde{d}(\bar{y}_\mu, (f, \varphi)(\bar{y}_\mu))} \right) + \gamma \left( \tilde{d}(\bar{x}_\lambda, \bar{y}_\mu) \right) \quad \text{for each} \quad \bar{x}_\lambda, \bar{y}_\mu \in \bar{X}, \bar{x}_\lambda \neq \bar{y}_\mu.
\]

For each \(\bar{x}_\lambda, \bar{y}_\mu \in \bar{X}, \bar{x}_\lambda \neq \bar{y}_\mu\), where \(\psi, \varphi\) are altering distance functions, and

\[
\tilde{M}(\bar{x}_\lambda, \bar{y}_\mu) = \alpha \left( \frac{\tilde{d}(\bar{x}_\lambda, (f, \varphi)(\bar{x}_\lambda)) + \tilde{d}(\bar{y}_\mu, (f, \varphi)(\bar{y}_\mu))}{1 + \tilde{d}(\bar{x}_\lambda, (f, \varphi)(\bar{x}_\lambda)) + \tilde{d}(\bar{y}_\mu, (f, \varphi)(\bar{y}_\mu))} \right) + \beta \left( \frac{\tilde{d}(\bar{x}_\lambda, (f, \varphi)(\bar{x}_\lambda)) + \tilde{d}(\bar{y}_\mu, (f, \varphi)(\bar{y}_\mu))}{1 + \tilde{d}(\bar{x}_\lambda, (f, \varphi)(\bar{x}_\lambda)) + \tilde{d}(\bar{y}_\mu, (f, \varphi)(\bar{y}_\mu))} \right) + \gamma \left( \tilde{d}(\bar{x}_\lambda, \bar{y}_\mu) \right) \quad \text{for each} \quad \bar{x}_\lambda, \bar{y}_\mu \in \bar{X}, \bar{x}_\lambda \neq \bar{y}_\mu.
\]

Where \(\alpha, \beta, \gamma > 0\) and \(2\alpha + 2\beta + \gamma < 1\) is a soft constant. Then \((f, \varphi)\) has a unique fixed point in \(\bar{X}\).

**Proof:** Let \(\bar{x}_\lambda^0\) be any soft point in \(SP(X)\).

Set

\[
\bar{x}_\lambda^1 = (f, \varphi)(\bar{x}_\lambda^0) = (f(\bar{x}_\lambda^0))_{\varphi(\lambda)}
\]

\[
\bar{x}_\lambda^2 = (f, \varphi)(\bar{x}_\lambda^1) = (f^2(\bar{x}_\lambda^0))_{\varphi^2(\lambda)}
\]

\[
\vdots
\]

\[
\bar{x}_\lambda^{n+1} = (f, \varphi)(\bar{x}_\lambda^n) = \left( f^{n+1}(\bar{x}_\lambda^0) \right)_{\varphi^{n+1}(\lambda)}, \ldots
\]

Now consider, from (3.1.2) we have

\[
\tilde{M}(\bar{x}_\lambda^{n-1}, \bar{x}_\lambda^n) = \alpha \left( \frac{\tilde{d}(\bar{x}_\lambda^{n-1}, (f, \varphi)(\bar{x}_\lambda^{n-1})) + \tilde{d}(\bar{x}_\lambda^n, (f, \varphi)(\bar{x}_\lambda^n))}{1 + \tilde{d}(\bar{x}_\lambda^{n-1}, (f, \varphi)(\bar{x}_\lambda^{n-1})) + \tilde{d}(\bar{x}_\lambda^n, (f, \varphi)(\bar{x}_\lambda^n))} \right) + \beta \left( \frac{\tilde{d}(\bar{x}_\lambda^{n-1}, (f, \varphi)(\bar{x}_\lambda^{n-1})) + \tilde{d}(\bar{x}_\lambda^n, (f, \varphi)(\bar{x}_\lambda^n))}{1 + \tilde{d}(\bar{x}_\lambda^{n-1}, (f, \varphi)(\bar{x}_\lambda^{n-1})) + \tilde{d}(\bar{x}_\lambda^n, (f, \varphi)(\bar{x}_\lambda^n))} \right) + \gamma \left( \tilde{d}(\bar{x}_\lambda^{n-1}, \bar{x}_\lambda^n) \right)
\]

\[
= \alpha \left( \frac{\tilde{d}(\bar{x}_\lambda^{n-1}, \bar{x}_\lambda^n) + \tilde{d}(\bar{x}_\lambda^n, \bar{x}_\lambda^{n+1})}{1 + \tilde{d}(\bar{x}_\lambda^{n-1}, \bar{x}_\lambda^n) + \tilde{d}(\bar{x}_\lambda^n, \bar{x}_\lambda^{n+1})} \right)
\]
From (3.1.1) we have

\[
\begin{align*}
\psi[d(\bar{x}_n^0, \bar{x}_{n+1}^0)] &= \psi\left[d\left((f_0, \varphi)(\bar{x}^n_{\alpha_{n-1}}), (f, \varphi)(\bar{x}^n_\alpha)\right]\right] \\
&\leq \psi\left[\varphi(\bar{x}^n_{\alpha_{n-1}}) - \varphi(\bar{x}^n_\alpha)\right] \\
&\leq \psi\left[\alpha + \beta + \gamma\right] d(\bar{x}^n_{\alpha_{n-1}}, \bar{x}^n_\alpha) + (\alpha + \beta) \varphi(\bar{x}^n_\alpha) \\
&\leq \psi\left[\alpha + \beta + \gamma\right] d(\bar{x}^n_{\alpha_{n-1}}, \bar{x}^n_\alpha) + (\alpha + \beta) \varphi(\bar{x}^n_\alpha) \\
&\leq \psi\left[\alpha + \beta + \gamma\right] d(\bar{x}^n_{\alpha_{n-1}}, \bar{x}^n_\alpha) + (\alpha + \beta) \varphi(\bar{x}^n_\alpha)
\end{align*}
\]

Since \(\psi\) is non-decreasing, we have

\[
\begin{align*}
d(\bar{x}^n_{\alpha_{n-1}}, \bar{x}^n_\alpha) &\leq (\alpha + \beta + \gamma) d(\bar{x}^{n-1}_{\alpha_{n-1}}, \bar{x}^n_\alpha) + (\alpha + \beta) d(\bar{x}^n_\alpha, \bar{x}^{n+1}_\alpha) \\
\tilde{d}(\bar{x}^n_{\alpha_{n-1}}, \bar{x}^n_\alpha) &\leq \frac{\alpha + \beta + \gamma}{(1-\alpha - \beta)} d(\bar{x}^{n-1}_{\alpha_{n-1}}, \bar{x}^n_\alpha) \\
\tilde{d}(\bar{x}^n_{\alpha_{n-1}}, \bar{x}^n_\alpha) &\leq h \tilde{d}(\bar{x}^{n-1}_{\alpha_{n-1}}, \bar{x}^n_\alpha) \\
\end{align*}
\]

Where \(h = \frac{\alpha + \beta + \gamma}{1-\alpha - \beta}\)

Thus \(\tilde{d}(\bar{x}^n_{\alpha_{n-1}}, \bar{x}^n_\alpha) \leq h^n \tilde{d}(\bar{x}^0_{\alpha_0}, \bar{x}^1_\alpha)\)

Taking \(n \to \infty\), we have

\[
\lim_{n \to \infty} \tilde{d}(\bar{x}^n_{\alpha_{n-1}}, \bar{x}^n_\alpha) = 0 \quad \text{...(3.1.3)}
\]
Now, we will show that \( \{\tilde{x}^n_{\lambda_n}\} \) is a soft Cauchy sequence. Suppose that \( \{\tilde{x}^n_{\lambda_m}\} \) is not a Soft Cauchy sequence, which means that there is a constant \( \epsilon_0 > 0 \) such that for each positive integer \( k \), there are positive integer \( \lambda_{m(k)} \) and \( \lambda_{n(k)} \) with \( \lambda_{m(k)} > \lambda_{n(k)} > k \) such that

\[
\tilde{d}\left(\tilde{x}^{m(k)}_{\lambda_{m(k)}}, \tilde{x}^{n(k)}_{\lambda_{n(k)}}\right) \geq \epsilon_0, \quad \tilde{d}\left(\tilde{x}^{m(k)}_{\lambda_{m(k)}}, \tilde{x}^{n(k)}_{\lambda_{n(k)}}\right) < \epsilon_0
\]

By triangle inequality

\[
\epsilon_0 \leq \tilde{d}\left(\tilde{x}^{m(k)}_{\lambda_{m(k)}}, \tilde{x}^{n(k)}_{\lambda_{n(k)}}\right) \leq \tilde{d}\left(\tilde{x}^{m(k)}_{\lambda_{m(k)}}, \tilde{x}^{m(k)-1}_{\lambda_{m(k)}}\right) + \tilde{d}\left(\tilde{x}^{m(k)-1}_{\lambda_{m(k)}}, \tilde{x}^{n(k)}_{\lambda_{n(k)}}\right) < \tilde{d}\left(\tilde{x}^{m(k)}_{\lambda_{m(k)}}, \tilde{x}^{m(k)-1}_{\lambda_{m(k)}}\right) + \epsilon_0
\]

Letting \( k \to \infty \) and using (3.1.3), we have

\[
\lim_{k \to \infty} \tilde{d}\left(\tilde{x}^{m(k)}_{\lambda_{m(k)}}, \tilde{x}^{n(k)}_{\lambda_{n(k)}}\right) = \epsilon_0 \quad \cdots (3.1.4)
\]

Similarly, we have

\[
\lim_{n \to \infty} \tilde{d}\left(\tilde{x}^{m(k)}_{\lambda_{m(k)}}, \tilde{x}^{n(k)+1}_{\lambda_{n(k)+1}}\right) = \epsilon_0, \quad \lim_{n \to \infty} \tilde{d}\left(\tilde{x}^{m(k)+1}_{\lambda_{m(k)+1}}, \tilde{x}^{n(k)}_{\lambda_{n(k)}}\right) = \epsilon_0, \quad \lim_{n \to \infty} \tilde{d}\left(\tilde{x}^{m(k)+1}_{\lambda_{m(k)+1}}, \tilde{x}^{n(k)+1}_{\lambda_{n(k)+1}}\right) = \epsilon_0. \quad \cdots (3.1.5)
\]

Putting \( \tilde{x}_\lambda = \tilde{x}^{m(k)}_{\lambda_{m(k)}} \) and \( \tilde{y}_\mu = \tilde{x}^{n(k)}_{\lambda_{n(k)}} \) in (3.1.2) we have

\[
\tilde{M}\left(\tilde{x}^{m(k)}_{\lambda_{m(k)}}, \tilde{x}^{n(k)}_{\lambda_{n(k)}}\right) = \alpha \frac{\tilde{d}\left(\tilde{x}^{m(k)}_{\lambda_{m(k)}}, \tilde{x}^{n(k)}_{\lambda_{n(k)}}\right)}{1 + \tilde{d}^2}\left(\tilde{x}^{m(k)}_{\lambda_{m(k)}}, \tilde{x}^{m(k)}_{\lambda_{m(k)}}\right) + \beta \frac{\tilde{d}\left(\tilde{x}^{n(k)}_{\lambda_{n(k)}}, \tilde{x}^{n(k)}_{\lambda_{n(k)}}\right)}{1 + \tilde{d}^2}\left(\tilde{x}^{n(k)}_{\lambda_{n(k)}}, \tilde{x}^{m(k)}_{\lambda_{m(k)}}\right) + \gamma \left(\tilde{d}\left(\tilde{x}^{m(k)}_{\lambda_{m(k)}}, \tilde{x}^{n(k)}_{\lambda_{n(k)}}\right)\right)
\]

\[
\tilde{M}\left(\tilde{x}^{m(k)}_{\lambda_{m(k)}}, \tilde{x}^{n(k)}_{\lambda_{n(k)}}\right) = \alpha \frac{\tilde{d}\left(\tilde{x}^{m(k)+1}_{\lambda_{m(k)+1}}, \tilde{x}^{n(k)+1}_{\lambda_{n(k)+1}}\right)}{1 + \tilde{d}^2}\left(\tilde{x}^{m(k)+1}_{\lambda_{m(k)+1}}, \tilde{x}^{m(k)+1}_{\lambda_{m(k)+1}}\right) + \beta \frac{\tilde{d}\left(\tilde{x}^{n(k)+1}_{\lambda_{n(k)+1}}, \tilde{x}^{n(k)+1}_{\lambda_{n(k)+1}}\right)}{1 + \tilde{d}^2}\left(\tilde{x}^{n(k)+1}_{\lambda_{n(k)+1}}, \tilde{x}^{m(k)+1}_{\lambda_{m(k)+1}}\right) + \gamma \left(\tilde{d}\left(\tilde{x}^{m(k)+1}_{\lambda_{m(k)+1}}, \tilde{x}^{n(k)+1}_{\lambda_{n(k)+1}}\right)\right)
\]
Letting \( k \to \infty \) and using (3.1.3), (3.1.4) and (3.1.5), we have

\[
\lim_{k \to \infty} \tilde{M}(\tilde{x}^m(k), \tilde{x}^n(k)) = (\beta + \gamma)\varepsilon_0 \quad \ldots (3.1.6)
\]

From (3.1.1) we have

\[
\psi \left[ \tilde{d} \left( \tilde{x}^m(k+1), \tilde{x}^n(k+1) \right) \right] = \psi \left[ \tilde{d} \left( f, \varphi \right) \left( \tilde{x}^m(k), \tilde{x}^n(k) \right) \right] 
\leq \psi \left[ \tilde{M} \left( \tilde{x}^m(k), \tilde{x}^n(k) \right) \right] - \varphi \left[ \tilde{M} \left( \tilde{x}^m(k), \tilde{x}^n(k) \right) \right]
\]

Taking \( k \to \infty \), using (3.1.5), (3.1.6) and the continuity of \( \psi \) and \( \varphi \), we have

\[
\psi[\varepsilon_0] \leq \psi[(\beta + \gamma)\varepsilon_0] - \varphi[(\beta + \gamma)\varepsilon_0] 
\leq \psi[\varepsilon_0] - \varphi[(\beta + \gamma)\varepsilon_0]
\]

This leads to \( \varphi[(\beta + \gamma)\varepsilon_0] = 0 \), and property of \( \varphi \) we get \( \varepsilon_0 = 0 \).

This is a contradiction. Thus \( \{\tilde{x}^n_\lambda\} \) is a soft Cauchy sequence in \( \tilde{X} \), which is complete. Thus, there is \( \tilde{x}_\lambda^* \in \tilde{X} \) such that \( \tilde{x}^n_\lambda \to \tilde{x}_\lambda^* \), \( n \to \infty \). \quad \ldots (3.1.6)

Putting \( \tilde{x}_\lambda = \tilde{x}^n_\lambda \) and \( \tilde{y}_\mu = \tilde{x}_\lambda^* \) in (3.1.2) we have

\[
\tilde{M} \left( \tilde{x}^n_\lambda, \tilde{x}_\lambda^* \right) = \alpha \left\{ \frac{\tilde{d} \left( \tilde{x}^n_\lambda(f, \varphi)(\tilde{x}^n_\lambda) \right) + \tilde{d} \left( \tilde{x}_\lambda^*(f, \varphi)(\tilde{x}_\lambda^*) \right)}{1 + \tilde{d} \left( \tilde{x}^n_\lambda(f, \varphi)(\tilde{x}_\lambda^*) \right) + \tilde{d} \left( \tilde{x}_\lambda^*(f, \varphi)(\tilde{x}_\lambda^*) \right)} \right\} 
\leq \alpha \left\{ \frac{\tilde{d} \left( \tilde{x}^n_\lambda(f, \varphi)(\tilde{x}^n_\lambda) \right) + \tilde{d} \left( \tilde{x}_\lambda^*(f, \varphi)(\tilde{x}_\lambda^*) \right)}{1 + \tilde{d} \left( \tilde{x}^n_\lambda(f, \varphi)(\tilde{x}_\lambda^*) \right) + \tilde{d} \left( \tilde{x}_\lambda^*(f, \varphi)(\tilde{x}_\lambda^*) \right)} \right\} + \beta \left\{ \frac{\tilde{d} \left( \tilde{x}^n_\lambda(f, \varphi)(\tilde{x}_\lambda^*) \right) + \tilde{d} \left( \tilde{x}_\lambda^*(f, \varphi)(\tilde{x}_\lambda^*) \right)}{1 + \tilde{d} \left( \tilde{x}^n_\lambda(f, \varphi)(\tilde{x}_\lambda^*) \right) + \tilde{d} \left( \tilde{x}_\lambda^*(f, \varphi)(\tilde{x}_\lambda^*) \right)} \right\} + \gamma \{\tilde{d}(\tilde{x}^n_\lambda, \tilde{x}_\lambda^*)\}
\]

Taking \( n \to \infty \) and using (3.1.3) and (3.1.6) we have

\[
\lim_{n \to \infty} \tilde{M} \left( \tilde{x}^n_\lambda, \tilde{x}_\lambda^* \right) \leq (\alpha + \beta)\{\tilde{d}(\tilde{x}_\lambda^*, (f, \varphi)(\tilde{x}_\lambda^*))\}
\]

From (3.1.1) we have

\[
\psi \left[ \tilde{d} \left( \tilde{x}^{n+1}_\lambda, (f, \varphi)(\tilde{x}_\lambda^*) \right) \right] = \psi \left[ \tilde{d} \left( (f, \varphi)(\tilde{x}^n_\lambda), (f, \varphi)(\tilde{x}_\lambda^*) \right) \right]
\]
Which implies
\[ \psi [\bar{d}(\bar{x}_\lambda^*, (f, \varphi)(\bar{x}_\lambda^*))] \leq \psi [(\alpha + \beta)\{\bar{d}(\bar{x}_\lambda^*, (f, \varphi)(\bar{x}_\lambda^*))\}] - \varphi[(\alpha + \beta)\{\bar{d}(\bar{x}_\lambda^*, (f, \varphi)(\bar{x}_\lambda^*))\}]
\]
\[ \psi [\bar{d}(\bar{x}_\lambda^*, (f, \varphi)(\bar{x}_\lambda^*))] \leq \psi [\bar{d}(\bar{x}_\lambda^*, (f, \varphi)(\bar{x}_\lambda^*))] - \varphi[(\alpha + \beta)\{\bar{d}(\bar{x}_\lambda^*, (f, \varphi)(\bar{x}_\lambda^*))\}]
\]

Which implies \( \varphi[(\alpha + \beta)\{\bar{d}(\bar{x}_\lambda^*, (f, \varphi)(\bar{x}_\lambda^*))\}] = 0, \)

So \( \bar{d}(\bar{x}_\lambda^*, (f, \varphi)(\bar{x}_\lambda^*)) = 0, \) that is \((f, \varphi)(\bar{x}_\lambda^*) = \bar{x}_\lambda^*.\)

**Uniqueness:** Let \( \bar{y}_\mu^* \) is another fixed point of \((f, \varphi)\) in \( \bar{X} \) such that \( \bar{x}_\lambda^* \neq \bar{y}_\mu^* \), then we have

Putting \( \bar{x}_\lambda = \bar{x}_\lambda^* \) and \( \bar{y}_\mu = \bar{y}_\mu^* \) in (3.1.2) we have

\[ \tilde{M} (\bar{x}_\lambda^*, \bar{y}_\mu^*) = \alpha \left\{ \frac{\bar{d}^2(\bar{x}_\lambda^*, (f, \varphi)(\bar{x}_\lambda^*)) + \bar{d}^2(\bar{y}_\mu^*, (f, \varphi)(\bar{y}_\mu^*))}{1 + \bar{d}^2(\bar{x}_\lambda^*, (f, \varphi)(\bar{x}_\lambda^*)) + \bar{d}^2(\bar{y}_\mu^*, (f, \varphi)(\bar{y}_\mu^*))} \right\} + \beta \left\{ \frac{\bar{d}^2(\bar{x}_\lambda^*, (f, \varphi)(\bar{y}_\mu^*)) + \bar{d}^2(\bar{y}_\mu^*, (f, \varphi)(\bar{x}_\lambda^*))}{1 + \bar{d}^2(\bar{x}_\lambda^*, (f, \varphi)(\bar{x}_\lambda^*)) + \bar{d}^2(\bar{y}_\mu^*, (f, \varphi)(\bar{y}_\mu^*))} \right\} + \gamma \{\bar{d}(\bar{x}_\lambda^*, \bar{y}_\mu^*)\}
\]

\[ \tilde{M} (\bar{x}_\lambda^*, \bar{y}_\mu^*) \leq (\beta + \gamma)\bar{d}(\bar{x}_\lambda^*, \bar{y}_\mu^*)
\]

From (3.1.1) we have

\[ \psi [\bar{d}(\bar{x}_\lambda^*, \bar{y}_\mu^*)] = \psi [\bar{d}(\bar{x}_\lambda^*, (f, \varphi)(\bar{x}_\lambda^*), (f, \varphi)(\bar{y}_\mu^*))]
\]

\[ \leq \psi [\tilde{M}(\bar{x}_\lambda^*, \bar{y}_\mu^*]) - \varphi[\tilde{M}(\bar{x}_\lambda^*, \bar{y}_\mu^*)]
\]

\[ \leq \psi [(\beta + \gamma)\bar{d}(\bar{x}_\lambda^*, \bar{y}_\mu^*)] - \varphi[(\beta + \gamma)\bar{d}(\bar{x}_\lambda^*, \bar{y}_\mu^*)]
\]

\[ \psi [\bar{d}(\bar{x}_\lambda^*, \bar{y}_\mu^*)] \leq \psi [\bar{d}(\bar{x}_\lambda^*, \bar{y}_\mu^*)] - \varphi[(\beta + \gamma)\bar{d}(\bar{x}_\lambda^*, \bar{y}_\mu^*)]
\]

So \( \varphi[(\beta + \gamma)\bar{d}(\bar{x}_\lambda^*, \bar{y}_\mu^*)] = 0, \) thus \( \bar{d}(\bar{x}_\lambda^*, \bar{y}_\mu^*) = 0, \) that is \( \bar{x}_\lambda^* = \bar{y}_\mu^*.\)

Hence fixed point of \((f, \varphi)\) is unique.

**Corollary 3.2:** Let \((\bar{X}, \bar{d}, E)\) be a soft complete metric space. Suppose the soft mapping \((f, \varphi): (\bar{X}, \bar{d}, E) \rightarrow (\bar{X}, \bar{d}, E)\) satisfies the following condition:

\[ \psi \left[ \bar{d} \left( (f, \varphi)(\bar{x}_\lambda), (f, \varphi)(\bar{y}_\mu) \right) \right] \leq \psi [\tilde{M}(\bar{x}_\lambda, \bar{y}_\mu)] - \varphi[\tilde{M}(\bar{x}_\lambda, \bar{y}_\mu)] \quad \ldots (3.2.1)
\]

For each \( \bar{x}_\lambda, \bar{y}_\mu \in \bar{X}, \bar{x}_\lambda \neq \bar{y}_\mu, \) where \( \psi, \varphi \) are altering distance functions, and
\[ M(\tilde{x}_\lambda, \tilde{y}_\mu) = \alpha \left\{ \tilde{d}^2(\tilde{x}_\lambda, (f, \varphi)(\tilde{x}_\lambda)) + \tilde{d}^2(\tilde{y}_\mu, (f, \varphi)(\tilde{y}_\mu)) \right\} \]

\[ + \beta \left\{ \tilde{d}^2(\tilde{x}_\lambda, (f, \varphi)(\tilde{y}_\mu)) + \tilde{d}^2(\tilde{y}_\mu, (f, \varphi)(\tilde{x}_\lambda)) \right\} \]

\[ + \gamma \tilde{d}(\tilde{x}_\lambda, \tilde{y}_\mu) \]

\[ + \delta \tilde{d}(\tilde{x}_\lambda, (f, \varphi)(\tilde{x}_\lambda)) + \tilde{d}(\tilde{y}_\mu, (f, \varphi)(\tilde{y}_\mu)) \]

\[ + \eta \tilde{d}(\tilde{x}_\lambda, (f, \varphi)(\tilde{y}_\mu)) + \tilde{d}(\tilde{y}_\mu, (f, \varphi)(\tilde{x}_\lambda)) \] \quad \ldots (3.2.2)

Where \( \alpha, \beta, \gamma, \delta, \eta > 0 \) and \( 2\alpha + 2\beta + \gamma + 2\delta + 2\eta < 1 \) is a soft constant. Then \((f, \varphi)\) has a unique fixed point in \( \tilde{X} \).

**Proof:** It can be proved easily.

**Theorem 3.3:** Let \((\tilde{X}, \tilde{d}, E)\) be a soft complete metric space. Suppose the soft mapping \((f, \varphi) : (\tilde{X}, \tilde{d}, E) \to (\tilde{X}, \tilde{d}, E)\) satisfies the following condition:

\[ \psi \left[ \tilde{d}(f, \varphi)(\tilde{x}_\lambda), (f, \varphi)(\tilde{y}_\mu) \right] \leq \psi[M(\tilde{x}_\lambda, \tilde{y}_\mu)] - \varphi[M(\tilde{x}_\lambda, \tilde{y}_\mu)] \] \quad \ldots (3.3.1)

For each \( \tilde{x}_\lambda, \tilde{y}_\mu \in \tilde{X}, \tilde{x}_\lambda \neq \tilde{y}_\mu \), where \( \psi, \varphi \) are altering distance functions, and

\[ \tilde{M}(\tilde{x}_\lambda, \tilde{y}_\mu) = \alpha \left\{ \tilde{d}^2(\tilde{x}_\lambda, (f, \varphi)(\tilde{x}_\lambda)) + \tilde{d}^2(\tilde{y}_\mu, (f, \varphi)(\tilde{y}_\mu)) + \tilde{d}^2(\tilde{y}_\mu, (f, \varphi)(\tilde{x}_\lambda)) \right\} \]

\[ + \beta \left\{ \tilde{d}^2(\tilde{x}_\lambda, (f, \varphi)(\tilde{y}_\mu)) + \tilde{d}^2(\tilde{y}_\mu, (f, \varphi)(\tilde{x}_\lambda)) + \tilde{d}^2(\tilde{x}_\lambda, \tilde{y}_\mu) \right\} \]

\[ + \gamma \tilde{d}(\tilde{x}_\lambda, \tilde{y}_\mu) \]

\[ + \delta \tilde{d}(\tilde{x}_\lambda, (f, \varphi)(\tilde{x}_\lambda)) + \tilde{d}(\tilde{y}_\mu, (f, \varphi)(\tilde{y}_\mu)) \]

\[ + \eta \tilde{d}(\tilde{x}_\lambda, (f, \varphi)(\tilde{y}_\mu)) + \tilde{d}(\tilde{y}_\mu, (f, \varphi)(\tilde{x}_\lambda)) \] \quad \ldots (3.3.2)

Where \( \alpha, \beta > 0 \) and \( 2\alpha + 3\beta < 1 \) is a soft constant. Then \((f, \varphi)\) has a unique fixed point in \( \tilde{X} \).

**Proof:** Let \( \tilde{x}_\lambda^0 \) be any soft point in \( SP(X) \).

Set

\[ \tilde{x}_{\lambda_1} = (f, \varphi)(\tilde{x}_\lambda^0) = \left( f(\tilde{x}_\lambda^0) \right)_{\varphi(\lambda)} \]

\[ \tilde{x}_{\lambda_2} = (f, \varphi)(\tilde{x}_{\lambda_1}) = \left( f^2(\tilde{x}_\lambda^0) \right)_{\varphi^2(\lambda)} \]

\[ \vdots \]

\[ \tilde{x}_{\lambda_{n+1}} = (f, \varphi)(\tilde{x}_{\lambda_n}) = \left( f^{n+1}(\tilde{x}_\lambda^0) \right)_{\varphi^{n+1}(\lambda)} \]
Now consider from (3.3.1) we have

\[ \widetilde{M}(\bar{x}_{\lambda_{n-1}}, \bar{x}_n) = \alpha \left\{ \frac{\bar{d}^{2}(\bar{x}_{\lambda_{n-1}}, f, \phi)(\bar{x}_{\lambda_{n-1}}) + \bar{d}^{2}(\bar{x}_n, f, \phi)(\bar{x}_n)}{d(\bar{x}_{\lambda_{n-1}}, f, \phi)(\bar{x}_{\lambda_{n-1}}) + \bar{d}(\bar{x}_n, f, \phi)(\bar{x}_n)} \right\} + \beta \left\{ \frac{\bar{d}^{2}(\bar{x}_{\lambda_{n-1}}, f, \phi)(\bar{x}_n) + \bar{d}^{2}(\bar{x}_{\lambda_{n-1}}, f, \phi)(\bar{x}_{\lambda_{n-1}})}{d(\bar{x}_{\lambda_{n-1}}, f, \phi)(\bar{x}_n) + \bar{d}(\bar{x}_{\lambda_{n-1}}, f, \phi)(\bar{x}_{\lambda_{n-1}})} \right\} \]

\[ = \alpha \left\{ \frac{\bar{d}^{2}(\bar{x}_{\lambda_{n-1}}, f, \phi)(\bar{x}_{\lambda_{n-1}}) + \bar{d}^{2}(\bar{x}_n, f, \phi)(\bar{x}_n)}{d(\bar{x}_{\lambda_{n-1}}, f, \phi)(\bar{x}_{\lambda_{n-1}}) + \bar{d}(\bar{x}_n, f, \phi)(\bar{x}_n)} \right\} + \beta \left\{ \frac{\bar{d}^{2}(\bar{x}_{\lambda_{n-1}}, f, \phi)(\bar{x}_n) + \bar{d}^{2}(\bar{x}_{\lambda_{n-1}}, f, \phi)(\bar{x}_{\lambda_{n-1}})}{d(\bar{x}_{\lambda_{n-1}}, f, \phi)(\bar{x}_n) + \bar{d}(\bar{x}_{\lambda_{n-1}}, f, \phi)(\bar{x}_{\lambda_{n-1}})} \right\} \]

\[ \leq \alpha \left\{ \frac{(\bar{d}(\bar{x}_{\lambda_{n-1}}, f, \phi)(\bar{x}_{\lambda_{n-1}}) + \bar{d}(\bar{x}_n, f, \phi)(\bar{x}_n))^{2}}{d(\bar{x}_{\lambda_{n-1}}, f, \phi)(\bar{x}_{\lambda_{n-1}}) + \bar{d}(\bar{x}_n, f, \phi)(\bar{x}_n)} \right\} + \beta \left\{ \frac{(\bar{d}(\bar{x}_{\lambda_{n-1}}, f, \phi)(\bar{x}_n) + \bar{d}(\bar{x}_{\lambda_{n-1}}, f, \phi)(\bar{x}_{\lambda_{n-1}}))^{2}}{d(\bar{x}_{\lambda_{n-1}}, f, \phi)(\bar{x}_n) + \bar{d}(\bar{x}_{\lambda_{n-1}}, f, \phi)(\bar{x}_{\lambda_{n-1}})} \right\} \]

\[ \leq \alpha \{d(\bar{x}_{\lambda_{n-1}}, \bar{x}_n) + d(\bar{x}_n, \bar{x}_{\lambda_{n+1}})\} + \beta \{d(\bar{x}_{\lambda_{n-1}}, \bar{x}_{\lambda_{n+1}}) + d(\bar{x}_n, \bar{x}_{\lambda_n})\} \]

\[ \leq (\alpha + 2\beta)\{d(\bar{x}_{\lambda_{n-1}}, \bar{x}_n)\} + (\alpha + \beta)\{d(\bar{x}_n, \bar{x}_{\lambda_{n+1}})\} \]

From (3.3.1), we have

\[ \psi(\bar{d}(\bar{x}_{\lambda_{n-1}}, \bar{x}_{\lambda_{n+1}})) = \psi\left[d\left((f, \phi)(\bar{x}_{\lambda_{n-1}}), (f, \phi)(\bar{x}_n)\right)\right] \]

\[ \leq \psi\left[\widetilde{M}(\bar{x}_{\lambda_{n-1}}, \bar{x}_n) - \phi\left[\widetilde{M}(\bar{x}_{\lambda_{n-1}}, \bar{x}_n)\right]\right] \]

\[ \leq \psi(\alpha + 2\beta)\{d(\bar{x}_{\lambda_{n-1}}, \bar{x}_n)\} + (\alpha + \beta)\{d(\bar{x}_n, \bar{x}_{\lambda_{n+1}})\} - \phi\left[\widetilde{M}(\bar{x}_{\lambda_{n-1}}, \bar{x}_n)\right] \]

\[ \psi(\bar{d}(\bar{x}_{\lambda_{n-1}}, \bar{x}_{\lambda_{n+1}})) \leq \psi(\alpha + 2\beta)\{d(\bar{x}_{\lambda_{n-1}}, \bar{x}_n)\} + (\alpha + \beta)\{d(\bar{x}_n, \bar{x}_{\lambda_{n+1}})\} \]

Since \(\psi\) is non-decreasing, we have

\[ \bar{d}(\bar{x}_{\lambda_{n-1}}, \bar{x}_{\lambda_{n+1}}) \leq (\alpha + 2\beta)\{d(\bar{x}_{\lambda_{n-1}}, \bar{x}_n)\} + (\alpha + \beta)\{d(\bar{x}_n, \bar{x}_{\lambda_{n+1}})\} \]

\[ \bar{d}(\bar{x}_{\lambda_{n-1}}, \bar{x}_{\lambda_{n+1}}) \leq \frac{(\alpha+2\beta)(\alpha-\beta)}{ \bar{d}(\bar{x}_{\lambda_{n-1}}, \bar{x}_n) \bar{d}(\bar{x}_n, \bar{x}_{\lambda_{n+1}}) } \]
\[ d(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}}) \leq h \cdot d(\tilde{x}^{n-1}_{\lambda_{n-1}}, \tilde{x}^n_{\lambda_n}) \]

Where \( h = \frac{(a+2\beta)}{1-\alpha-\beta} \)

Thus \( d(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}}) \leq h^n d(\tilde{x}^0_{\lambda_0}, \tilde{x}^1_{\lambda_1}) \)

Taking \( n \to \infty \), we have

\[ \lim_{n \to \infty} d(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}}) = 0 \] \( \ldots(3.3.3) \)

Now, we will show that \( \{\tilde{x}^n_{\lambda_n}\} \) is a soft Cauchy sequence. Suppose that \( \{\tilde{x}^n_{\lambda_n}\} \) is not a Soft Cauchy sequence, which means that there is a constant \( \varepsilon_0 > 0 \) such that for each positive integer \( k \), there are positive integer \( \lambda_{m(k)} \) and \( \lambda_{n(k)} \) with \( \lambda_{m(k)} > \lambda_{n(k)} > k \) such that

\[ d\left(\tilde{x}^{m(k)}_{\lambda_{m(k)}}, \tilde{x}^{n(k)}_{\lambda_{n(k)}}\right) \geq \varepsilon_0, d\left(\tilde{x}^{m(k)-1}_{\lambda_{m(k)-1}} , \tilde{x}^{n(k)}_{\lambda_{n(k)}}\right) < \varepsilon_0 \]

By triangle inequality

\[ \varepsilon_0 \leq d\left(\tilde{x}^{m(k)}_{\lambda_{m(k)}}, \tilde{x}^{n(k)}_{\lambda_{n(k)}}\right) \leq d\left(\tilde{x}^{m(k)}_{\lambda_{m(k)}}, \tilde{x}^{m(k)-1}_{\lambda_{m(k)-1}}\right) + d\left(\tilde{x}^{m(k)-1}_{\lambda_{m(k)-1}}, \tilde{x}^{n(k)}_{\lambda_{n(k)}}\right) \]

\[ < d\left(\tilde{x}^{m(k)}_{\lambda_{m(k)}}, \tilde{x}^{m(k)-1}_{\lambda_{m(k)-1}}\right) + \varepsilon_0 \]

Letting \( k \to \infty \) and using \( (3.3.3) \), we have

\[ \lim_{n \to \infty} d\left(\tilde{x}^{m(k)}_{\lambda_{m(k)}}, \tilde{x}^{n(k)}_{\lambda_{n(k)}}\right) = \varepsilon_0 \] \( \ldots(3.3.4) \)

Similarly, we have

\[ \begin{align*}
\lim_{n \to \infty} \tilde{d}\left(\tilde{x}^{m(k)}_{\lambda_{m(k)}}, \tilde{x}^{n(k)+1}_{\lambda_{n(k)+1}}\right) &= \varepsilon_0, \\
\lim_{n \to \infty} \tilde{d}\left(\tilde{x}^{m(k)}_{\lambda_{m(k)}}, \tilde{x}^{n(k)}_{\lambda_{n(k)}}\right) &= \varepsilon_0, \\
\lim_{n \to \infty} \tilde{d}\left(\tilde{x}^{m(k)+1}_{\lambda_{m(k)+1}}, \tilde{x}^{n(k)+1}_{\lambda_{n(k)+1}}\right) &= \varepsilon_0.
\end{align*} \] \( \ldots(3.3.5) \)

Putting \( \tilde{x}_\lambda = \tilde{x}^{m(k)}_{\lambda_{m(k)}} \) and \( \tilde{y}_\mu = \tilde{x}^{n(k)}_{\lambda_{n(k)}} \) in \( (3.3.2) \) we have

\[ \tilde{N}\left(\tilde{x}^{m(k)}_{\lambda_{m(k)}}, \tilde{x}^{n(k)}_{\lambda_{n(k)}}\right) = \alpha \left\{ \begin{array}{c}
\tilde{d}\left(\tilde{x}^{m(k)}_{\lambda_{m(k)}}, \tilde{x}^{m(k)}_{\lambda_{m(k)}}\right) + \tilde{d}\left(\tilde{x}^{n(k)}_{\lambda_{n(k)}}, \tilde{x}^{n(k)}_{\lambda_{n(k)}}\right) + \tilde{d}\left(\tilde{y}_\mu, \tilde{x}^{m(k)}_{\lambda_{m(k)}}\right) \\
\tilde{d}\left(\tilde{x}^{m(k)}_{\lambda_{m(k)}}, \tilde{x}^{m(k)}_{\lambda_{m(k)}}\right) + \tilde{d}\left(\tilde{x}^{n(k)}_{\lambda_{n(k)}}, \tilde{x}^{n(k)}_{\lambda_{n(k)}}\right) + \tilde{d}\left(\tilde{y}_\mu, \tilde{x}^{m(k)}_{\lambda_{m(k)}}\right)
\end{array} \right\} \]
Letting and using (3.3.3) and (3.3.5), we have
\[
\lim_{k \to \infty} \mathcal{M} \left( \tilde{x}_{\lambda(m)}^{m(k)}, \tilde{x}_{\lambda(n)}^{n(k)} \right) = (\alpha + \beta) \epsilon_0 \tag{3.3.6}
\]
From (3.3.1) we have
\[
\psi \left[ \tilde{d} \left( \tilde{x}_{\lambda(m(k)+1)}^{m(k)+1}, \tilde{x}_{\lambda(n(k)+1)}^{n(k)+1} \right) \right] = \psi \left[ \tilde{d} \left( (f, \varphi) \left( \tilde{x}_{\lambda(m(k))}^{m(k)}, (f, \varphi) \left( \tilde{x}_{\lambda(n(k))}^{n(k)} \right) \right) \right]
\leq \psi \left[ \mathcal{M} \left( \tilde{x}_{\lambda(m(k))}^{m(k)}, \tilde{x}_{\lambda(n(k))}^{n(k)} \right) \right] - \varphi \left[ \mathcal{M} \left( \tilde{x}_{\lambda(m(k))}^{m(k)}, \tilde{x}_{\lambda(n(k))}^{n(k)} \right) \right]
\]
Taking \( k \to \infty \) and from (3.3.5), (3.3.6), we have
\[
\psi[\epsilon_0] \leq \psi[(\alpha + \beta) \epsilon_0] - \varphi[(\alpha + \beta) \epsilon_0]
\leq \psi[\epsilon_0] - \varphi[(\alpha + \beta) \epsilon_0]
\]
This leads to \( \varphi[(\alpha + \beta) \epsilon_0] = 0 \), and property of \( \varphi \) we get \( \epsilon_0 = 0 \).

This is a contradiction. Thus \( \{\tilde{x}_{\lambda(n)}^{n}\} \) is a soft Cauchy sequence, which is complete. Thus, there is \( \tilde{x}_\lambda^* \in \tilde{X} \) such that \( \tilde{x}_{\lambda(n)}^{n} \to \tilde{x}_\lambda^* \), \( n \to \infty \). \( \tag{3.3.7} \)

Putting \( \tilde{x}_\lambda = \tilde{x}_{\lambda(n)}^{n} \) and \( \tilde{y}_\mu = \tilde{x}_\lambda^* \) in (3.3.2) we have
\[
\mathcal{M} \left( \tilde{x}_{\lambda(n)}, \tilde{x}_\lambda^* \right) = \alpha \left[ \frac{d^2 (f, \varphi) (\tilde{x}_{\lambda(n)}^{n})}{d (\tilde{x}_{\lambda(n)}^{n}, (f, \varphi) (\tilde{x}_{\lambda(n)}^{n}))} + \frac{d^2 (f, \varphi) (\tilde{x}_{\lambda(n)}^{n})}{d (\tilde{x}_{\lambda(n)}^{n}, (f, \varphi) (\tilde{x}_{\lambda(n)}^{n}))} + \frac{d^2 (f, \varphi) (\tilde{x}_{\lambda(n)}^{n})}{d (\tilde{x}_{\lambda(n)}^{n}, (f, \varphi) (\tilde{x}_{\lambda(n)}^{n}))} + \frac{d^2 (f, \varphi) (\tilde{x}_{\lambda(n)}^{n})}{d (\tilde{x}_{\lambda(n)}^{n}, (f, \varphi) (\tilde{x}_{\lambda(n)}^{n}))} \right]
\]
\[
+ \beta \left\{ \frac{d^2 (f, \varphi) (\tilde{x}_\lambda^*)}{d (\tilde{x}_{\lambda(n)}^{n}, (f, \varphi) (\tilde{x}_{\lambda(n)}^{n}))} + \frac{d^2 (f, \varphi) (\tilde{x}_\lambda^*)}{d (\tilde{x}_{\lambda(n)}^{n}, (f, \varphi) (\tilde{x}_{\lambda(n)}^{n}))} + \frac{d^2 (f, \varphi) (\tilde{x}_\lambda^*)}{d (\tilde{x}_{\lambda(n)}^{n}, (f, \varphi) (\tilde{x}_{\lambda(n)}^{n}))} + \frac{d^2 (f, \varphi) (\tilde{x}_\lambda^*)}{d (\tilde{x}_{\lambda(n)}^{n}, (f, \varphi) (\tilde{x}_{\lambda(n)}^{n}))} \right\}
\]
Taking $n \to \infty$ and using (3.3.3), (3.3.7) we have

$$\lim_{n \to \infty} \tilde{M}(\tilde{x}_{\lambda_n}^n, \tilde{x}_{\lambda}^*) = (\alpha + \beta)\{\tilde{d}(\tilde{x}_{\lambda}^*, (f, \varphi)(\tilde{x}_{\lambda}^*))\}$$

From (3.3.1) we have

$$\psi[\tilde{d}(\tilde{x}_{\lambda_n}^{n+1}, (f, \varphi)(\tilde{x}_{\lambda}^*))] = \psi[\tilde{d}(\tilde{x}_{\lambda_n}^n, (f, \varphi)(\tilde{x}_{\lambda}^*))]$$

$$\leq \psi[\tilde{M}(\tilde{x}_{\lambda_n}^n, \tilde{x}_{\lambda}^*)] - \varphi[\tilde{d}(\tilde{x}_{\lambda_n}^n, \tilde{x}_{\lambda}^*)]$$

$$\psi[\tilde{d}(\tilde{x}_{\lambda}^*, (f, \varphi)(\tilde{x}_{\lambda}^*))] \leq \psi[(\alpha + \beta)\{\tilde{d}(\tilde{x}_{\lambda}^*, (f, \varphi)(\tilde{x}_{\lambda}^*))\}] - \varphi[(\alpha + \beta)\{\tilde{d}(\tilde{x}_{\lambda}^*, (f, \varphi)(\tilde{x}_{\lambda}^*))\}]$$

$$\psi[\tilde{d}(\tilde{x}_{\lambda}^*, (f, \varphi)(\tilde{x}_{\lambda}^*))] \leq \psi[\tilde{d}(\tilde{x}_{\lambda}^*, (f, \varphi)(\tilde{x}_{\lambda}^*))] - \varphi[(\alpha + \beta)\{\tilde{d}(\tilde{x}_{\lambda}^*, (f, \varphi)(\tilde{x}_{\lambda}^*))\}]$$

Which implies $\varphi[(\alpha + \beta)\{\tilde{d}(\tilde{x}_{\lambda}^*, (f, \varphi)(\tilde{x}_{\lambda}^*))\}] = 0$,

So $\tilde{d}(\tilde{x}_{\lambda}^*, (f, \varphi)(\tilde{x}_{\lambda}^*)) = 0$, that is $(f, \varphi)(\tilde{x}_{\lambda}^*) = \tilde{x}_{\lambda}^*$.

**Uniqueness:** Let $\tilde{y}_{\mu}^*$ is another fixed point of $(f, \varphi)$ in $\tilde{X}$ such that $\tilde{x}_{\lambda}^* \neq \tilde{y}_{\mu}^*$, then we have

Putting $\tilde{x}_{\lambda} = \tilde{x}_{\lambda}^*$ and $\tilde{y}_{\mu} = \tilde{y}_{\mu}^*$ in (3.3.2) we have

$$\tilde{M}(\tilde{x}_{\lambda}^*, \tilde{y}_{\mu}^*) = \alpha \left\{ \frac{\tilde{d}^2(\tilde{x}_{\lambda}^*, (f, \varphi)(\tilde{x}_{\lambda}^*)) + \tilde{d}^2(\tilde{y}_{\mu}^*, (f, \varphi)(\tilde{y}_{\mu}^*)) + \tilde{d}^2(\tilde{y}_{\mu}^*, (f, \varphi)(\tilde{x}_{\lambda}^*))}{\tilde{d}(\tilde{x}_{\lambda}^*, (f, \varphi)(\tilde{x}_{\lambda}^*)) + \tilde{d}(\tilde{y}_{\mu}^*, (f, \varphi)(\tilde{y}_{\mu}^*)) + \tilde{d}(\tilde{y}_{\mu}^*, (f, \varphi)(\tilde{x}_{\lambda}^*))} \right\}$$

$$+ \beta \left\{ \frac{\tilde{d}^2(\tilde{x}_{\lambda}^*, (f, \varphi)(\tilde{x}_{\lambda}^*)) + \tilde{d}^2(\tilde{y}_{\mu}^*, (f, \varphi)(\tilde{y}_{\mu}^*)) + \tilde{d}^2(\tilde{y}_{\mu}^*, \tilde{y}_{\mu}^*)}{\tilde{d}(\tilde{x}_{\lambda}^*, (f, \varphi)(\tilde{x}_{\lambda}^*)) + \tilde{d}(\tilde{y}_{\mu}^*, (f, \varphi)(\tilde{y}_{\mu}^*)) + \tilde{d}(\tilde{y}_{\mu}^*, \tilde{y}_{\mu}^*)} \right\}$$

$$\tilde{M}(\tilde{x}_{\lambda}^*, \tilde{y}_{\mu}^*) = (\alpha + \beta)\{\tilde{d}(\tilde{x}_{\lambda}^*, \tilde{y}_{\mu}^*)\}$$

From (3.3.1) we have

$$\psi[\tilde{d}(\tilde{x}_{\lambda}^*, \tilde{y}_{\mu}^*)] = \psi[\tilde{d}(\tilde{x}_{\lambda}^*, (f, \varphi)(\tilde{x}_{\lambda}^*))]$$

$$\leq \psi[\tilde{M}(\tilde{x}_{\lambda}^*, \tilde{y}_{\mu}^*)] - \varphi[\tilde{M}(\tilde{x}_{\lambda}^*, \tilde{y}_{\mu}^*)]$$

$$\leq \psi[(\alpha + \beta)\tilde{d}(\tilde{x}_{\lambda}^*, \tilde{y}_{\mu}^*)] - \varphi[(\alpha + \beta)\tilde{d}(\tilde{x}_{\lambda}^*, \tilde{y}_{\mu}^*)]$$

$$\psi[\tilde{d}(\tilde{x}_{\lambda}^*, \tilde{y}_{\mu}^*)] \leq \psi[\tilde{d}(\tilde{x}_{\lambda}^*, \tilde{y}_{\mu}^*)] - \varphi[(\alpha + \beta)\tilde{d}(\tilde{x}_{\lambda}^*, \tilde{y}_{\mu}^*)]$$

So $\varphi[(\alpha + \beta)\tilde{d}(\tilde{x}_{\lambda}^*, \tilde{y}_{\mu}^*)] = 0$, thus $\tilde{d}(\tilde{x}_{\lambda}^*, \tilde{y}_{\mu}^*) = 0$, that is $\tilde{x}_{\lambda}^* = \tilde{y}_{\mu}^*$. 
Hence fixed point of \((f, \varphi)\) is unique.

References