Fractals: Fractional Dimension and Self-Similarity

Sumita Singh

Abstract—Fractals are objects that have fractional dimension and reveal self similarity in structure at all scales. A snowflake curve can be produced by starting with an equilateral triangle and dividing each side into three equal segments. They give an insight into the self-similarity-organizations of things in nature. So Fractals can be defined as geometric figures, just like circles, rectangles and squares, but fractals have specific properties that those figures do not have. Algebraically fractals are the result of repetitions of nonlinear equations. The most important properties of fractals are self-similarity and non-integer dimension. Hence Fractal Dimension allows us to measure the complexity of an object. Fractals improved our precision in describing and classifying "random" or organic objects, but maybe they are not perfect.

Keywords—self-similarity, cantor dust, koch curve, fractal geometry.

I. INTRODUCTION

Fractals are objects that have fractional dimension and reveal self similarity in structure at all scales. A snowflake curve can be produced by starting with an equilateral triangle and dividing each side into three equal segments. They give an insight into the self-similarity-organizations of things in nature. So Fractals can be defined as geometric figures, just like circles, rectangles and squares, but fractals have specific properties that those figures do not have. Algebraically fractals are the result of repetitions of nonlinear equations. The most important properties of fractals are self-similarity and non-integer dimension. Therefore Fractal Dimension allows us to measure the complexity of an object. In real world they are present everywhere in nature. For example considering the river and its tributaries, each tributary has its own tributaries leading to a structure which is similar to that of the entire river. In fractal geometry they each have a simple organizing principle. This idea of trying to find underlying theories in system having seemingly random variations is called the chaos theory. This theory is applied in order to study weather patterns, the stock market and population dynamics.

Fractal distributions are hierarchical, like smoke trails or billowy clouds in the sky. Turbulence shapes both the clouds in the sky and the clouds in space, giving them an irregular but repetitive pattern that would be impossible to describe without the help of fractal geometry. Fractal have significant role to play in astrophysical world.

Fractal Dimension allows us to measure the degree of complexity by evaluating how fast our measurements increase or decrease as our scale becomes larger or smaller and Fractals often look like objects in nature. A fractal is generated by a process involving successive subdivision such that it preserves its self-similar structure at all levels. To illustrate this we consider the rule “Trisect a line segment (say, initially of unit length) and remove its middle part repeatedly.” This yields the Cantor dust, given;

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<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>N(l)</th>
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<tbody>
<tr>
<td>0</td>
<td>1/3</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1/9</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>1/27</td>
<td>8</td>
</tr>
</tbody>
</table>

Figure 1: Cantor Dust
At each step we find that the length of segment reduces to one third of its previous value and number of segments is inversely proportional to the length of segment in each step. But power laws like $y = x^2$, $y=1$ etc. are considered to be the seeds of self-similarity. So we let,

$$\text{Number of segments} = \alpha \left( \frac{1}{\text{length of segment}} \right)^p$$

The value of power “$p$” gives the fractional dimension.

II. KOCH CURVE

A snowflake curve can be produced by starting with an equilateral triangle and dividing each side into three equal segments. The middle segments are then replaced by two equal segments which form the sides of a smaller equilateral triangle. This gives a 12-sided star-shaped figure. The next stage is to subdivide each of the sides of this figure in the same way, and so on. This results in a figure that resembles a snowflake. This is called Koch Curve. In the limit, this figure has, fractional dimension, $D_F = 1.26$.

![Figure 2: Koch Curve with $D_F = 1.26$](image)

III. FEATURES OF FRACTALS

Fractals are fascinating and important because of their peculiar properties. For example, Koch Curve has an infinite length but covers only a finite area. Sierpinski Carpet has infinite length but zero area. Menger Sponge with infinite surface area has zero volume.

IV. DISCUSSIONS

A line segment is divided into “$n$” equal parts. Then we can either remove or replace a segment. Suppose we replace a segment by $m$ equal segments which form the sides of an $(m+1)$-sided regular polygon. We observe following two parts-

1. For different values of $m$, we get different values of fractal dimension, $D_F$.
2. The value of $D_F$ does not depend upon which segment is getting replaced, provided $m$ be kept fixed.

*Note:* If one wish, he can replace some or all the parts of the line segment to have self-similarity and bearing fractal dimensions.

V. RULE

Divide a line segment into $n$ parts in the ratios $x_1 : x_2 : \ldots : x_n$ and then replace some or all the segments by $m$ equal respective segments such that they form the sides of $(m+1)$ sided regular polygons. Then find the value of $D_F$.

VI. APPLICATION

The study of self-similarity in figures is used in certain branches of physics, such as crystal growth. Fractals are also important in chaos theory and in computer graphics. Nature is abundant with objects having fractal dimensions. A few examples are borders of geographical masses like seashores, boundary of a nation, outline of cloud, cracks in earth etc.

VII. CONCLUSION

Many objects in nature aren't formed of squares or triangles, but of more complicated geometric figures. The implication of the brief discussion of fractal aesthetics is that the beneficial effects of contact with nature could be tapped without the presence of actual representations of nature, but with the fractal geometry that is characteristic of natural elements. Fractal geometry relates to structures that cannot be described by Euclidean whole number dimensions of 1 (straight lines), 2 (flat surfaces) or 3 (volumes).

Fractals improved our precision in describing and classifying "random" or organic objects, but maybe they are not perfect. Maybe they are just closer to our natural world, not the same as it. It is believed that true randomness does exist, and no mathematical equation will ever describe it perfectly.
Perhaps for many people think fractals will never represent anything more than beautiful pictures. Actually, the main property of fractals is their regularity among the seemingly incontrollable complexity and chaos. Fractal geometry provides a powerful approach for the quantitative description of complex, highly irregular and random, i.e. disordered systems. Moreover, it can be used to describe the processes leading to the formation of such systems and their physical behavior.

REFERENCES