A Model for Anisotropic Fluid Sphere.
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Solutions of Einstein’s field equations for spherically symmetric matter distribution have been solved by Adler[1], Whitman[6], Singh and Yadav [5], Yadav and Saini[7] and Yadav with others[8] using different methods and assumptions. The matter distribution is usually assumed to be locally isotropic, in case of pressure for massive objects in general relativity. Else from last some years theoretical studies on realistic stellar models indicate that some massive objects may be locally anisotropic. A number of interesting solutions that have provided insight into the effects of anisotropy on star parameters [2, 3].

Exact analytical solutions Einstein’s field equations are of much value in general relativity. These solutions are generally obtained by using different conditions and assumptions. One of the assumptions made for obtaining the solutions is that the space-time be conformally flat. This assumption has been widely used in relativity theory.

In this paper we have found an exact analytical solution of Einstein’s field equations for static anistropic fluid sphere by assuming that space-time is conformally flat and by taking a suitable form of metric potential. The model is physically reasonable and free from singularities. Energy density $\rho$, radial and tangential pressures have been calculated for the model.

I. THE FIELD EQUATIONS

We take spherically symmetric line element

$$ds^2 = e^\alpha dt^2 - e^\beta dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\Phi^2)$$

Where $\alpha$ and $\beta$ are functions of $r$ only.

The Einstein’s field equations

$$R^i_j - \frac{1}{2} R \delta^i_j = -8\pi T^i_j$$

For the spherically symmetric line element (2.1) yield

$$-8\pi T^1_1 = e^{-\alpha} \left( \frac{\beta'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2}$$

$$-8\pi T^2_2 = -8\pi T^3_3
= e^{-\alpha} \left( \frac{\beta'}{2} + \frac{\alpha' \beta'}{4} + \frac{\beta'^2}{2} + \frac{\beta' - \alpha'}{2r} \right)$$

$$8\pi T^4_4 = e^{-\alpha} \left( \frac{\alpha'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2}$$

Where a prime denotes differentiation with respect to $r$.

Throughout the investigation we set velocity of light $c$ and gravitational constant $k$ to be unity. The energy momentum tensor $T^i_j$ is given by

$$T^i_j = (\rho + p) u^i u_{j-p} \delta^i_j$$
For anisotropic fluid sphere the field equations given above go to the form

$$-8\pi \rho = e^{-\alpha} \left( \frac{1}{r^2} - \frac{\alpha'}{r} \right) - \frac{1}{r^2}$$  \hspace{1cm} (2.7)

$$-8\pi P_r = \frac{1}{r^2} - e^{-\alpha} \left( \frac{1}{r^2} + \frac{\alpha'}{r} \right)$$  \hspace{1cm} (2.8)

$$-8\pi P_{\perp r} = e^{-\alpha} \left[ \frac{\beta' \alpha'}{4} - \frac{\beta'^2}{4} - \frac{\beta'}{2} - \left( \frac{\beta' - \alpha'}{2r} \right) \right]$$  \hspace{1cm} (2.9)

Where $\rho$ is energy density and $P_r$ and $P_{\perp r}$ are the radial and tangential pressure respectively.

We suppose that the space-time is conformally flat for which vanishing of Weyl tensor gives

$$\frac{\alpha''}{\alpha'} + \frac{1}{r^2} - \frac{\beta'^2}{4} - \frac{\beta'}{2} + \frac{1}{2r}(\beta' - \alpha') = 0$$  \hspace{1cm} (2.10)

Now we make use of the transformations

$$\exp(-\alpha) = \zeta$$  \hspace{1cm} (2.11)

$$\exp(\beta) = \eta^2$$  \hspace{1cm} (2.12)

$$x = r^2$$  \hspace{1cm} (2.13)

$$x\zeta, x + 1 - \zeta - 4\pi xD = 0$$  \hspace{1cm} (2.14)

$$\left(4\zeta x^2\right)\eta, xx + (2x^2\zeta, x - \zeta + 1)\eta = 0$$  \hspace{1cm} (2.15)

Where $D = P_r - P_{\perp r}$ and the subscript $x$ following a comma denotes differentiation with respect to $x$. Equations (2.14) and (2.15) on integration yield

$$\zeta = e^{-\alpha} = 1 + \alpha\mu r^2 + 8\pi r^2 \int_0^r \frac{(P_r - P_{\perp r})}{r} dr$$  \hspace{1cm} (2.16)

$$\eta^2 = e^\beta = r^2 \left[ Ae^{F(r)} + Be^{-F(r)} \right]^2$$  \hspace{1cm} (2.17)

Where $\mu, A$ and $B$ are constants of integration and

$$F(r) = \int \frac{\zeta}{r} dr$$  \hspace{1cm} (2.18)

The integration constants $\mu, A$ and $B$ can be evaluated by matching the metric functions given by (2.16) and (2.17) to the exterior Schwarzschild solution for a mass $m$ and radius $r_0$ as

$$A = \frac{F(r_0)}{2r_0} \left[ \left( 1 - \frac{2m}{r_0} \right)^{3/2} + \frac{3m}{r_0} - 1 \right]$$  \hspace{1cm} (2.19)

$$B = \frac{F(r_0)}{2r_0} \left[ \left( 1 - \frac{2m}{r_0} \right)^{3/2} - \frac{3m}{r_0} + 1 \right]$$  \hspace{1cm} (2.20)
II. SOLUTIONS OF FIELD EQUATIONS

We see that as a matter of fact equations (2.7) – (2.9) and (2.16) – (2.18) are three equations in four unknowns \( \rho, p_r, p_\perp r \) and \( F(r) \). Thus the system is indeterminate. To make the system determinate, we choose

\[
\sqrt{\xi} = e^{-\alpha/2} = \frac{(1+Dr^2)}{(1-Er^2)}
\]

Where D and E are constants. \( e^\beta, P_r, P_\perp r \) and \( \rho \) can be found from field equations and using (2.15)-(2.21). However, to simplify mathematics, we take \( D = \mu = E / 3 \) which yields the solution as

\[
e^\alpha/2 = \frac{1+3G\epsilon}{1-G\epsilon} = 1 + \frac{4G\epsilon}{1-G\epsilon}
\]

\[
e^\beta/2 = \frac{4G\epsilon(1-G)^4+(1+G)(1-3G)(1-G)^4}{(1-G)(1+3G)(1-G\epsilon)^2}
\]

\[
8\pi \rho r_0^2 = \frac{8G(3+2G\epsilon+3G^2\epsilon^2)}{(1+3G\epsilon)}
\]

\[
8\pi P_r r_0^2 = \frac{16G}{4G[(1-G^4)/(1-3G)^4(1-G)1+1(1-2G\epsilon+3G^2\epsilon^2)(1-G^4)]}
\]

\[
8\pi P_\perp r_0^2 = 8\pi P_r r_0^2 + \frac{16G^2\epsilon(5+3G\epsilon)}{(1-3G\epsilon)^3}
\]

Where \( \epsilon = \frac{r^2}{r_0^2} \)

\[
G = \frac{1-(1-2\xi)^{1/2}}{1+3(1-2n)^{1/2}}
\]

\[\xi = m/r_0\]

III. CONCLUSION

1. If we choose the equation of state \( P_r = P_\perp r \) then in this case we obtain the well-known Schwarzschild interior solution [4].

2. It is remarkable that the solution of Einstein’s field equation obtained here is singularity free and the density of fluid sphere drops continuously from its maximum value at the centre to the value which is positive at the boundary.

REFERENCES